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*By*

**R. KUMAR**

Author of Calculus, Algebra, Differential Equations, Three Dimensional Geometry,
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## PREFACE TO THE NEW MILLENNIUM EDITION

It gives me a great pleasure to offer the New Millennium Edition of the book to the readers. The entire book has been rewritten to incorporate new problems and new theory. The proofs of many theorems and examples have been simplified.

The subject matter of Real Analysis involves more of conceptual skill than computational skill. Unless the concepts and the related results are clear, one cannot possibly grasp Real Analysis. The book is written keeping these points in mind. The basic concepts and theorems of Real Analysis are given in a very simple, direct and easy to understand language so that the students are able to grasp these in the very first reading of the book. A large number of examples and counter-examples are given after these concepts. One of the chief features of the book is the clear cut answer to the statements of the form :

1. What happens if a set is **not** bounded above or below ?
2. What happens if a set is **not** open or closed ?
3. What happens if a real number is **not** a limit point of a set ?
4. What happens if a function is **not** continuous or uniformly continuous ? etc.

The author feels that the lack of understanding of such statements make the study of Real Analysis a nightmare.

Other prominent features of the book are :

1. A large number of well explained solved examples—including those asked in the latest university papers.
2. The proofs of the theorems are given in a very simple manner.
3. The topic of limit superior and limit inferior of a sequence explained in a very simple way.
4. Special emphasis on Sequences and Infinite Series.
5. Tricky problems of uniform continuity solved by easy methods.

I am confident that the new edition of the book will prove more useful and meet the requirements and challenges of the 21st Century.

**Author**

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# SYLLABUS

## B.A./B.Sc. (Hons.) I Year

### Analysis I (Unit 3)

**Section 1.** The real number system as a complete ordered field. Neighbourhoods, open and closed sets. Limit points of sets, Bolzano-Weierstrass theorem.

**Section 2.** Sequences, Convergent sequences, Cauchy sequences, Monotonic sequences, Subsequences, Limit superior and limit inferior of a sequence.

**Section 3.** Infinite series, Convergence of infinite series, Positive term series, Comparison tests, Cauchy's nth root test. D'Alembert's ratio test. Raabe's test, Cauchy-Maclaurin integral test, Alternating series, Leibnitz test. Absolute and conditional convergence.

**Section 4.** Successive Differentiation, Leibnitz Theorem. Partial differentiation, Curvature, Asymptotes, Singular points, Concavity, Convexity, Points of inflexion, Tracing of curves in cartesian and polar co-ordinates.

### Analysis II (Unit 4)

**Sections 1.** Limits, Continuity, Sequential Continuity, Algebra of continuous functions, Continuity of composite functions, continuity on $[a, b]$ implying boundedness, Intermediate Value Theorem, Inverse Function Theorem, Uniform continuity.

**Section 2.** Differentiation, Algebra of derivatives, Differentiability and continuity, Chain rule, Inverse function theorem, Darboux theorem, Rolle's theorem, Mean Value theorems, Taylor's theorem.

**Section 3.** Taylor's series, Maclaurin's series, Expansions of

$$\sin x, \cos x, e^x, \log(1+x), (1+x)^m.$$

Applications of mean value theorems to monotone functions and inequalities, Maxima and Minima, Indeterminate forms.

**Section 4.** Integration of irrational functions, Reduction formulae, Rectification, Quadrature, Volumes and Surfaces of revolution.

## B.A. (Pass)/B.Sc. (General) III Year

**Section 1.** The real number system as a complete ordered field. Neighbourhoods, Open and closed sets. Limit points of sets. Bolzano-Weierstrass theorem.

**Section 2.** Properties of continuous functions. Uniform continuity.

**Section 3.** Sequences, Convergent sequences, Cauchy sequences. Limit superior and limit inferior of a sequence.

**Section 4.** Infinite series and their convergence. Positive term series. Comparison tests, Cauchy's $n$th root test. D'Alembert's ratio tests, Raabe's test. Cauchy-Maclaurin integral test, Alternating series, Leibnitz's test, Absolute and conditional convergence.

**Section 5.** Rolle's Theorem, Mean Value Theorems, Taylor's Theorem with Lagrange's and Cauchy's forms of remainder. Taylor's series. Maclaurin's series expansions of $\sin x, \cos x, e^x, \log(1+x), (1+x)^m$.

**Section 6.** Applications of Mean Value theorems to monotonic functions and inequalities. Maxima and minima. Indeterminate forms.

# SOME IMPORTANT RESULTS OF REAL ANALYSIS

## SETS OF REAL NUMBERS

**1.** $|xy| = |x||y|, |x/y| = |x|/|y| \; (y \neq 0)$ ; $x, y \in \mathbf{R}$.

**2.** $|x+y| \leq |x| + |y|, \quad x, y \in \mathbf{R}$.

**3.** $||x| - |y|| \leq |x-y|$ ; $x, y \in \mathbf{R}$.

**4.** $|x-a| < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon \Leftrightarrow x \in ]a-\varepsilon, a+\varepsilon[$.

**5.** $|a| = \max\{a, -a\}, |a|^2 = a^2$.

**6.** **(Order-Completeness Property or R)**

Every non-empty and bounded above set of real numbers has a supremum in **R**.

**7.** The set **Q** of all rational numbers is not order-complete.

**8.** **(Archimedean Property)**

If $a$ and $b$ are any two positive real numbers, then there exists a positive integer $n$ such that $na > b$.

**9.** **(Open and Closed Sets)**

(*i*) The union of an arbitrary family of open sets is an open set.

(*ii*) The intersection of a finite number of open sets is an open set.

(*iii*) The union of a finite number of closed sets is a closed set.

(*iv*) The intersection of an arbitrary family of closed sets is a closed set.

**10.** **(Bolzano-Weierstrass Theorem)**

Every infinite bounded set has a limit point.

**11.** A set is closed iff it contains all its limit points.

**12.** A point $p$ is a limit point of a set $S$ iff every neighbourhood of $p$ contains infinitely many points of $S$.

## SEQUENCES

**13.** Limit of a sequence, if it exists, is unique.

**14.** Every convergent sequence is bounded but the converse is not true.

**15.** $\lim\limits_{n \to \infty} n^{1/n} = 1$.

**16.** $\lim\limits_{n \to \infty} r^n = 0$, if $|r| < 1$.

**17.** **(Cauchy's Principle of Convergence)**

A necessary and sufficient condition for the convergence of a sequence $< a_n >$ is that for each $\varepsilon > 0$, there exists a positive integer $m$ such that $|a_n - a_m| < \varepsilon \;\; \forall \; n \geq m$.

**18.** **(Cauchy's first theorem on limits)**

If $\lim\limits_{n \to \infty} a_n = l$, then $\lim\limits_{n \to \infty} \left( \dfrac{a_1 + a_2 + \ldots\ldots + a_n}{n} \right) = l$.

**19.** If $< a_n >$ be a sequence such that $\lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} = l$, where $|l| < 1$,

then $\lim\limits_{n \to \infty} a_n = 0$.

**20.** $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$, where $e = 2.7183......$

**21.** (*i*) A monotonically increasing and bounded above sequence is convergent.

(*ii*) A monotonically decreasing and bounded below sequence is convergent.

(*iii*) A monotonically increasing sequence which is not bounded above, diverges to $+\infty$.

**22.** A real number $l$ is a limit point of a sequence $< a_n >$ iff there exists a subsequence of $< a_n >$ converging to $l$.

**23.** A bounded sequence $< a_n >$ converges to $l$ if and only if

$$\lim_{n \to \infty} \sup a_n = \lim_{n \to \infty} \inf a_n = l$$

## INFINITE SERIES

**24.** The geometric series $1 + r + r^2 + r^3 + .... (r > 0)$ converges if $r < 1$ and diverges to $+\infty$ if $r \geq 1$.

**25.** The series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

**26.** **(First Comparison Test)**

Let $\Sigma u_n$ and $\Sigma v_n$ be two positive term series such that

$$u_n \leq k v_n \quad \forall \quad n \geq m.$$

($k$ is a fixed positive number and $m$ is a positive integer)

Then $\Sigma v_n$ converges $\Rightarrow \Sigma u_n$ converges,

and $\Sigma u_n$ diverges $\Rightarrow \Sigma v_n$ diverges.

**27.** **(Limit Form Test)**

Let $\Sigma u_n$ and $\Sigma v_n$ be two positive term series such that

$$\lim_{n \to \infty} \frac{u_n}{v_n} = l, \text{ where } l \neq 0 \text{ and } l \text{ is finite.}$$

Then $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together.

**28.** **(Cauchy's nth Root Test)**

If $\Sigma u_n$ is a positive term series such that $\lim\limits_{n \to \infty} (u_n)^{1/n} = l$, then $\Sigma u_n$ converges if $l < 1$ and $\Sigma u_n$ diverges if $l > 1$.

**29.** **(D' Alembert's Ratio Test)**

If $\Sigma u_n$ is a positive term series such that $\lim\limits_{n \to \infty} \frac{u_n}{u_{n+1}} = l$, then $\Sigma u_n$ converges if $l > 1$ and $\Sigma u_n$ diverges if $l < 1$.

Further, $\Sigma u_n$ is convergent if $l = \infty$.

**30.** **(Raabe's Test)**

If $\Sigma u_n$ is a positive term series such that

$$\lim_{n \to \infty} n \left( \frac{u_n}{u_{n+1}} - 1 \right) = l,$$

then $\Sigma u_n$ converges if $l > l$ and $\Sigma u_n$ diverges if $l < 1$.

**31.** **(Cauchy's Integral Test)**

If $u(x)$ is a non-negative, monotonically decreasing and integrable function such that $u(n) = u_n \ \forall \ n \in \mathbf{N}$, then

$$\sum_{n=1}^{\infty} u_n \quad \text{and} \int_1^{\infty} u(x)\, dx \text{ converge or diverge together.}$$

**32.** **(Leibnitz's Test)**

If the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} u_n$ is such that

(*i*) $u_{n+1} \leq u_n \ \forall \ n \in \mathbf{N}$ and (*ii*) $\lim\limits_{n \to \infty} u_n = 0$,

then the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} u_n$ converges.

**33.** Every absolutely convergent series is convergent, but the converse is not true.

## CONTINUOUS AND DERIVABLE FUNCTIONS

**34.** A function $f$ defined on an interval I is continuous at a point $p \in$ I iff for every sequence $< p_n >$ in I converging to $p$ implies $< f(p_n) >$ converges to $f(p)$.

**35.** If a function $f$ is continuous in a closed and bounded interval $[a, b]$, then $f$ is bounded in $[a, b]$ and attains its bounds in $[a, b]$.

**36.** If a function $f$ is continuous in $[a, b]$ and $f(a), f(b)$ are of opposite signs, then there exists some point $c \in ]a, b[$ such that $f(c) = 0$.

**37.** **(Intermediate Value Theorem)**

If a function $f$ is continuous in $[a, b]$ and $f(a) \neq f(b)$, then $f$ assumes every value between $f(a)$ and $f(b)$.

**38.** Every uniformly continuous function on an interval I is continuous on I, but the converse is not true.

39. If a function $f$ is continuous on a closed and bounded interval $[a, b]$, then it is uniformly continuous on $[a, b]$.

40. If a function is derivable at a point, then it is continuous at that point. The converse need not be true.

41. **(Rolle's Theorem)**
If a function $f$ is (*i*) continuous on $[a, b]$, (*ii*) derivable on$]a, b[$, and (*iii*) $f(a) = f(b)$, then there exists some point $c \in ]a, b[$ such that $f'(c) = 0$

42. **(Lagrange's Mean Value Theorem)**
If a function $f$ is (*i*) continuous on $[a, b]$, (*ii*) derivable on $]a, b[$, then there exists some point $c \in ]a, b[$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

43. (*i*) If $f'(x) > 0 \ \forall \ x \in I$, then $f$ is *strictly increasing* in $I$.
(*ii*) If $f'(x) < 0 \ \forall \ x \in I$, then $f$ is *strictly decreasing* in $I$.

44. **(Taylor's Theorem)**
If a function $f$ defined on $[a, a+h]$ is such that the $(n-1)$th derivative $f^{n-1}$ is confitnuous on $[a, a+h]$ and derivable on $]a, a+h[$, then there exists some $\theta$, $0 < \theta < 1$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2\,!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)\,!} f^{n-1}(a) + R_n,$$

where $R_n = \dfrac{h^n}{n\,!} f^n(a + \theta h)$, (Lagrange's Remainder)

or $R_n = \dfrac{h^n (1-\theta)^{n-1}}{(n-1)\,!} f^n(a + \theta h)$. (Cauchy's Remainder)

45. **(Maclaurin's Infinite Series)**
If $f$ possesses derivatives of every order in $[0, h]$ and $\lim\limits_{n \to \infty} R_n = 0$, then $\forall \ x \in [0, h]$,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2\,!} f''(0) + \frac{x^3}{3\,!} f'''(0) + \ldots\ldots, \text{ where}$$

$$R_n = \frac{x^n}{n\,!} f^n(\theta x),\ 0 < \theta < 1, \text{ (Lagrange's Remainder)}$$

$$R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)\,!} f^n(\theta x),\ (0 < \theta < 1). \text{ (Cauchy's Remainder)}$$

## MAXIMA AND MINIMA

46. If a function $f$ has an extreme value at a point c, then $f'(c) = 0$, provided $f'(c)$ exists.

47. (*i*) If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a ***maximum*** at $x = c$.
(*ii*) If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a ***minimum*** at $x = c$.
(*iii*) If $f'(c) = f''(c) = 0$ and $f'''(c) \neq 0$, then $f$ has neither maximum nor minimum at $x = c$.

## GUIDELINES IN ANALYSIS I (UNIT 3) FOR B.A./B.Sc. (Hons.) I year

**Statement of the properties of real numbers as an ordered field. Definition of Modulus. Properties of Moduli such as**

(*i*) $|xy| = |x||y|$, (*ii*) $|x+y| \leq |x| + |y|$

(*iii*) $||x| - |y|| \leq |x - y|$, (*iv*) $|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$

(*v*) $|x - y| < \varepsilon$ iff $y - \varepsilon < x < y + \varepsilon$.

Bounded above and bounded below sets, upper and lower bounds, the supremum and the infimum and their uniqueness.

**Theorem :** A real number $K$ is the supremum of a set $A$ if and only if (*i*) $x \leq K$, $\forall \ x \in A$ (*ii*) For each $\varepsilon > 0, \exists \ y \in A$ such that $K - \varepsilon < y$.

The corresponding theorem for the infimum.

The set of real numbers as a complete ordered field (in terms of the supremum); the equivalent statement in terms of the infimum; the Archimedean property of real numbers.

Definition of a neighbourhood. $A$ is a neighbourhood of $x$ if there exists some $\varepsilon > 0$ such that $x \in ]x - \varepsilon, x + \varepsilon[ \subset A$.

**Theorem :** A is a neighbourhood of $x$ if and only if there exists some open interval $(a, b)$ such that $x \in \ ]a, b[ \subset A$.

Examples of neighbourhoods. The following theorems :

1. The intersection of any two (a finite number of) neighbourhoods is a neighbourhood.

2. Superset of a neighbourhood is a neighbourhood.

Definition of an open set as a set which is a neighbourhood of each of its points; equivalently, $G$ is an open set if for each $x \in G$ there exists an $\varepsilon > 0$ such that $x \in \ ]x - \varepsilon, x + \varepsilon[ \subset G$ (or $x \in \ ]a,b[ \subset G$).

**Theorem :** The union of an arbitrary family of open sets is an open set.

**Theorem :** The intersection of two (finitely many) open sets is an open set. Examples to show that the intersection of an arbitrary family of open sets is not open. A closed set defined as the complement of an open set.

**Theorem :** The intersection of an arbitrary family of closed sets is closed.

**Theorem :** The union of a finite family of closed sets is closed.

A limit point of a set $A$ defined as : a point $x$ is a limit point of $A$ if for every $\varepsilon > 0$, $]x - \varepsilon, x + \varepsilon[$ contains a point of $A$ other than $x$. Equivalently, a point $x$ is a limit point of $A$ if every neighbourhood of $x$ contains a point of $A$ other than $x$.

**Theorem :** A point $x$ is a limit point of $A$ if and only if for each $\varepsilon > 0$, $]x - \varepsilon, x + \varepsilon[$ contains infinitely many elements of $A$.

Various illustrations including the set of limit points of

(*i*) The set of natural numbers (*ii*) The set of integers (*iii*) Any interval $]a, b[$, $]a, b[$ etc. (*iv*) The set of rationals (*v*) The set of reals (*vi*) Any finite set (*vii*) Empty set (*viii*) The set $\{1/n\}$, (*ix*) The set $\{(n+1)/n\}$.

lzano-Weierstrass Theorem. Examples to illustrate that conditions in pothesis of Bolzano-Weierstrass theorem cannot be relaxed.

**Theorem :** A set is closed if and only if it contains its limit points.

Definition of a sequence and of a convergent sequence.

**Theorem :** Limit of a sequence, if it exists, is unique.

Definitions of divergent sequences $(\langle l_n \rangle \to +\infty$, or $\langle l_n \rangle \to -\infty$

Oscillatory sequences. Definition of a bounded sequence.

**Theorem :** Every convergent sequence is bounded.

A bounded sequence may not be convergent.

(*i*) If $\langle a_n \rangle$, $\langle b_n \rangle$ are convergent sequences, then $\langle a_n + b_n \rangle$ and $\langle a_n b_n \rangle$ are convergent and $\text{Lt}\,(a_n + b_n) = \text{Lt}\,a_n + \text{Lt}\,b_n$ and $\text{Lt}\,(a_n b_n) = (\text{Lt}\,a_n)\,(\text{Lt}\,b_n)$. (*ii*) If $\langle a_n \rangle \to l$ and $l \neq 0$, then $\exists\, K > 0$ and $n_0$ such that

$$|a_n| \geq K \;\; \forall \;\; n \geq n_0.$$

(*iii*) If $\langle a_n \rangle \to l$ and $l \neq 0$ and $a_n \neq 0$ for every $n$, then $\left(\dfrac{1}{a_n}\right) \to \dfrac{1}{l}$.

(*iv*) If $\langle a_n \rangle \to l$ and $a_n \geq 0 \;\forall\; n$, then $l \geq 0$.

(*v*) If $\langle a_n \rangle \to l$, then $\left\langle \dfrac{a_1 + a_2 + \ldots + a_n}{n} \right\rangle \to l$

(*vi*) If $\langle a_n \rangle \to l$ and $l > 0$ and $a_n > 0 \;\forall\; n$, then $(a_1 a_2 \ldots a_n)^{1/n} > \to l$.

(*vii*) If $b_n \leq a_n \leq c_n$ and $\langle b_n \rangle \to l$ and $\langle c_n \rangle \to l$, then $\langle a_n \rangle \to l$.

(*viii*) If for a sequence $\langle a_n \rangle$, $\left\langle \dfrac{a_{n+1}}{a_n} \right\rangle \to l$ and $|l| < 1$, then $\langle a_n \rangle \to 0$.

The definition of a subsequence.

**Theorem :** If a sequence $\langle a_n \rangle$ converges to $l$, then every sub-sequence of $\langle a_n \rangle$ converges to $l$.

**Theorem :** Every bounded sequence has a convergent subsequence. The limit points (cluster points) of a sequence.

**Theorem :** $p$ is a limit point of a sequence $\langle a_n \rangle$ if and only if there exists a subsequence of $\langle a_n \rangle$ converging to $p$.

**Theorem :** The set of limit points of a bounded sequence is bounded.

**Theorem :** If $\langle a_n \rangle \to l$, then $l$ is the only limit point of $\langle a_n \rangle$. The definition of a Cauchy sequence with examples.

**Theorem :** Every Cauchy sequence is bounded. A bounded sequence may not be Cauchy. The Cauchy convergence criterion, applications.

Monotonic sequences, increasing and decreasing sequences.

**Theorem :** Every monotonically increasing sequence bounded above (resp. monotonically decreasing sequence bounded below, or monotonic bounded sequence) is convergent.

**Definition** of the limit superior and the limit inferior as the supermum and the infimum of the set of its limit points.

**Theorem :** For a bounded sequence $\langle a_n \rangle$, $l = \text{Lim Sup}\; a_n$ iff for every $\varepsilon > 0$, (*i*) $a_n > l - \varepsilon$ for infinitely many values of $n$.

(*ii*) $a_n < l + \varepsilon$ for all except finitely many values of $n$.

The corresponding theorem for the limit inferior

**Theorem :** A bounded sequence $\langle a_n \rangle \to l$ if and only if $\text{Lim Sup}\; a_n = \text{Lim Inf}\; a_n = l$ or equivalently, $l$ is the only limit point of $\langle a_n \rangle$.

Infinite series : sequences of partial sums, convergent, divergent and oscillating series, Cauchy's general principle of convergence for a series.

**Theorem :** A necessary condition for a series $\Sigma\, u_n$ to be convergent is that $\lim\limits_{n \to \infty} u_n = 0$. Examples to show that the condition is not sufficient.

**Series of positive terms.** Convergence of $\Sigma 1/n^p$.

Comparison tests, Cauchy's *n*th root test, D'Alembert's ratio test. Raabe's test. Integral test without proof. (*Questions based upon these tests only.*) Series of arbitrary terms, Leibniz test for alternating series. Absolutely convergent series.

**Theorem :** An absolutely convergent series is convergent.

Examples of conditionally convergent series.

## GUIDELINES IN ANALYSIS II (UNIT 4) FOR
## B.A./B.Sc. (Hons.) I Year

Limit of functions, Uniqueness of the limit, one-sided limits.

**Theorem.** $\lim\limits_{x \to c} f(x) = l$ if and only if $\lim\limits_{x \to c+0} f(x) = \lim\limits_{x \to c-0} f(x) = l$.

**Theorem.** $\lim\limits_{x \to c} f(x) = l$ if and only if for every sequence $\langle x_n \rangle$ in the domain of $f$ satisfying $\langle x_n \rangle \to c$, we have $\langle f(x_n) \rangle \to l$.

Algebra of limits. Definitions of

$$\lim_{x \to c+0} f(x) = \pm\infty,\; \lim_{x \to c-0} f(x) = \pm\infty,\; \lim_{x \to c} f(x) = \pm\infty$$

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Inverse function Theorem for continuous functions (to be stated only).

Uniform continuity, definition and examples, uniform continuity implies continuity but the converse is not true.

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The derivative of a function at a point and in an interval. Algebra of derivatives, Derivability implies continuity but the converse is not true. Chain rule.

Darboux Theorem and Inverse Function Theorem (to be stated only).

**Mean Value Theorems :**

Rolle's Theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, geometrical interpretation of these theorems.

Taylor's theorem with Lagrange's and Cauchy's form of remainders.

Taylor's and Maclaurin's series.

Expansion of the following functions in powers of $x$ :

$$e^x, \sin x, \cos x, (1+x)^m, \log(1+x).$$

Applications of mean value theorems to obtain inequalities.

**Maxima and Minima :**

The extreme values of a function, the maximum and the minimum values of a function.

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First derivative test, Second derivative test.

General $n$th derivative test (to be stated only).

L'Hospital rule (without proof) for determining the following indeterminate forms :

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0.\infty, 0^0, \infty^0, 1^\infty.$$

Integration of functions of the following types

$$\sqrt{ax^2+bx+c}, \quad \frac{1}{\sqrt{ax^2+bx+c}}, \quad \frac{ax+b}{\sqrt{px^2+qx+r}}$$

$$(ax+b)\sqrt{px^2+qx+r}, \quad \frac{1}{(ax+b)\sqrt{px+r}}, \quad \frac{1}{(ax^2+b)\sqrt{px^2+q}}.$$

Integration of $\sin^n x$, $\cos^n x$, $\sin^n x \cos^m x$.

Reduction formulae for the integrals of the type

$$\int (a^2+x^2)^{n/2} dx, \int x^m \sqrt{2ax-x^2}\, dx.$$

Evaluation of

$$\int_0^{\pi/2} \sin^2 x\, dx, \quad \int_0^{\pi/2} \cos^n x\, dx, \quad \int_0^{\pi/2} \sin^n x \cos^m x\, dx.$$

**Rectification, Quadrature, Volume and surface of revolution (about axis only) of simple standard curves.**

# 1

# Real Numbers

In this chapter we shall introduce the *set of real numbers* and give its structure as a *complete ordered field*. Then we discuss the *Archimedean property of real numbers* and the *modulus of a real number* alongwith several properties. We begin this chapter with some set-theoretic concepts.

## 1.1 Sets

***A* set *is a well defined collection of objects.*** The objects of the set are called *elements* or *members* of the set. Capital letters $X, Y, S, T$ etc. generally denote sets, while small letters $x, y, s, t$ etc. denote elements of a set. If $x$ is an element of a set $S$, then we write $x \in S$ and read it as '$x$ belongs to $S$' or '$x$ is a member of $S$'. If $x$ is not an element of a set $S$, then we write $x \notin S$ and read it as '$x$ does not belong to $S$' or '$x$ is not a member of $S$'.

*A set having no element is called an* **empty set** *or a* **null set**. It is denoted by $\phi$.

*A* **non-empty set** $S$, *written as* $S \neq \phi$, *is a set which has at least one element.*

*A set is said to be* **finite** or **infinite** *according as it has finite or infinite number of elements.*

**Illustrations**

1. $\mathbf{N} = \{1, 2, 3, 4, \ldots\}$ is the set of natural numbers.
   Here $0 \notin \mathbf{N}$.
2. $\mathbf{I} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of integers.
3. The set $S = \{11, 12, 13, 14, \ldots\}$ may be written as
   $$S = \{x : x \in \mathbf{N} \text{ and } x > 10\}.$$
4. The set $T = \{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\}$ may be written as
   $$T = \{5x : x \in \mathbf{I}\}.$$
5. The set $S = \{x : x \in \mathbf{N} \text{ and } 2 < x < 3\}$ is an empty set *i.e.*, $S = \phi$.

## 1.2 Some Standard Notations

Some standard notations such as

$$\Rightarrow, \quad \Leftrightarrow, \quad \forall, \quad \exists, \quad \wedge, \quad \vee, \quad \varepsilon \quad \text{and} \quad \delta$$

will be frequently used in the book. Let us explain these.

($a$) $\Rightarrow$ means '*implies that*'

For example, $x = 2 \Rightarrow x^2 = 4.$

$x = 3 \Rightarrow x$ is an odd integer.

ESUT =

leton Set :- A set consisting of a single element is called a singleton set.

{ }

{1,2,3,4}

efinition 4. *The* **intersection** *of two sets S and T, denoted as* *is defined as*

$$S \cap T = \{x : x \in S \text{ and } x \in T\}.$$

= {1,2,3 other words, $S \cap T$ contains the common elements of $S$ and $T$.

**efinition 5.** *Two sets S and T are said to be* **disjoint** *if* $S \cap T = \phi$.

**tions** If $A \cap B \neq \phi$ then A and B are said to be interchanging or overlapping sets.

t $S = \{1, 3, 5\}$, $T = \{2, 3, 4, 5, 7\}$.

ESnT: en $S \cup T = \{1, 2, 3, 4, 5, 7\}$ and $S \cap T = \{3, 5\}$.

t $E = \{..., -6, -4, -2, 0, 2, 4, 6, ...\}$,

T = { } $O = \{... -5, -3, -1, 1, 3, 5, ...\}$.

en $E \cup O = \mathbf{I}$ (set of all integers)

y empty d $E \cap O = \phi$. Thus $E$ and $O$ are disjoint sets.

ey set **te.** It will help to remember the following results :

$$\boxed{\begin{aligned} x \in S \cup T &\Rightarrow x \in S \text{ or } x \in T.\\ x \in S \cap T &\Rightarrow x \in S \text{ and } x \in T.\\ x \notin S \cup T &\Rightarrow x \notin S \text{ and } x \notin T.\\ x \notin S \cap T &\Rightarrow x \notin S \text{ or } x \notin T.\end{aligned}}$$

**Definition 6.** *A non-empty set Δ is said to be an* **index set** *for the family* $\mathfrak{I} = \{S_\lambda\}$ *of sets if for each* $\lambda \in \Delta$, *there exists a set* $S_\lambda$ *in the family* $\mathfrak{I}$.

For example, $\mathbf{N} = \{1, 2, 3, 4, ...., ..\}$ is an index set for the family of sets $\{S_1, S_2, S_3, S_4, ...\}$.

Consider an arbitrary family of sets $\{S_\lambda : \lambda \in \Delta\}$, Δ is any index set.

**Definition 7.** *The* **union** *of the sets* $\{S_\lambda\}$ *is defined as*

$$\bigcup_{\lambda \in \Delta} S_\lambda = \{x : x \in S_\lambda \text{ for some } \lambda \in \Delta\}.$$

**Definition 8.** *The* **intersection** *of the sets* $\{S_\lambda\}$ *is defined as*

$$\bigcap_{\lambda \in \Delta} S_\lambda = \{x : x \in S_\lambda \text{ for each } \lambda \in \Delta\}.$$

**Definition 9.** *If S and T are two sets, then the* **difference** *of S and T, denoted as* $S \sim T$, *is defined as*

$$S \sim T = \{x : x \in S \text{ and } x \notin T\}.$$

For example, if $S = \{1, 2, 3, 4, 8\}$ and $T = \{2, 3, 5, 6, 7\}$, then

$$S \sim T = \{1, 4, 8\}.$$

**Definition 10.** *If S is a subset of a set X, then the* **complement** *of S, denoted* by $X \sim S$, *is defined as*

$$X \sim S = \{x : x \in X \text{ and } x \notin S\}.$$

*Other notations of the complement of S are* $S'$ *or* $S^c$.

Thus $x \in X \sim S \Rightarrow x \in X$ but $x \notin S$.

For example, if $X = \{1, 2, 3, 4, ..., 10\}$, and $S = \{2, 4, 6, 8, 10\}$, then $X \sim S = \{1, 3, 5, 7, 9\}$.

1) domain 4) Image
codomain 5) Pre-image

3) Range

$f : S \to T$

Raj is a brother of Ashok
R = B(A)

## 1.4 Function or Mapping

Domain :- which all values S can take
Range :- Represented by T

**Definition 1.** *If S and T are two non-empty sets, then a* **function** *from S to T is a rule, denoted as $f : S \to T$, which associates to each element $x \in S$, a unique element $y \in T$. The element y, written as $y = f(x)$, is called the* **image** *of x and x is called a* **pre-image** *of y. The set S is called the* **domain** *and T the* **co-domain** *of the function $f : S \to T$.*

*The set $f(S) = \{f(x) : x \in S\}$ is called the* **range** *of the function $f : S \to T$.*

Every Element of S (say x) is Uniquely associated to element of T (say y)

**Illustrations**

1. $f : \mathbf{N} \to \mathbf{N}$ defined by $f(n) = n^2$ is a function whose range is $\{1, 4, 9, 16, 25, \ldots\ldots\}$.
2. $f : \mathbf{N} \to \mathbf{I}$ defined by $f(n) = (-1)^n$ is a function whose **range** is $\{-1, 1\}$.
3. $f(x) = \sqrt{a^2 - x^2}$, $-a \le x \le a$, is a function whose domain is $\{x : -a \le x \le a\}$ and range is the set of positive real numbers.

**Definition 2.** *A mapping $f : S \to T$ is called* **one-to-one,** *if*

$$x \neq y \quad \Rightarrow \quad f(x) \neq f(y) \ ; \ x, y \in S.$$

*Or*

$$f(x) = f(y) \ \Rightarrow \quad x = y \quad ; \ x, y \in S.$$

**Definition 3.** *A mapping $f : S \to T$ is called* **onto**, *if for each $t \in T$, there exists some $s \in S$ such that $f(s) = t$.*

**Definition 4.** *Two mappings $f : S \to T$ and $g : S \to T$ are said to be* **equal**, *if*

$$f(x) = g(x) \ \ \forall \ x \in S.$$

**Definition 5.** *The* **product** *or the* **composite** *of two mappings*

$$f : S \to T \quad \text{and} \quad g : T \to U$$

*is defined as the mapping $gof : S \to U$ such that*

$$(gof)(x) = g(f(x)) \ \ \forall \ \ x \in S.$$

**Illustrations**

1. $f : \mathbf{N} \to \mathbf{N}$ defined as $f(x) = x^2 \ \forall \ x \in \mathbf{N}$ is *one-to-one*, since for any $x, y \in \mathbf{N}$,

   $$x \neq y \ \Rightarrow \ x^2 \neq y^2 \ \Rightarrow \ f(x) \neq f(y).$$

   However, $f$ is **not** onto ; since for $2 \in \mathbf{N}$, there does not exist any $x \in \mathbf{N}$ such that $f(x) = 2$. (Notice that $x^2 = 2 \Rightarrow x = \pm\sqrt{2} \notin \mathbf{N}$)
2. $f : \mathbf{I} \to \mathbf{I}$ defined as $f(x) = x + 3 \ \forall \ x \in \mathbf{I}$ is one-to-one and onto. Let $x, y \in \mathbf{I}$. Then $f(x) = f(y) \Rightarrow x + 3 = y + 3 \Rightarrow x = y$ and so $f$ is one-to-one. Let $x \in \mathbf{I}$ be arbitrary. Then $y = x - 3 \in \mathbf{I}$ and $f(y) = x - 3 + 3 = x$ and so $f$ is onto.
3. $f : \mathbf{I} \to \mathbf{I}$ defined by $f(x) = x^2 \ \forall \ x \in \mathbf{I}$, is **not** one-to-one. Notice that $2 \neq -2$ but $f(2) = f(-2) = 4$. Also $f$ is **not** onto.
4. Let $f : \mathbf{N} \to \mathbf{N}$ and $g : \mathbf{N} \to \mathbf{N}$ be defined as

Rational Numbers:- Numbers which can be expressed in the form $\frac{x}{y}$ where x and y are integers and $y \neq 0$ are known as rational numbers. These have a terminating or recurring decimal representation.

**Example 2.** *Prove that* $\sqrt{8}$ *is not a rational number.*

**Solution.** Let, if possible, $\sqrt{8}$ be a rational number.

Then $\sqrt{8} = \frac{p}{q}$; where $p, q \in \mathbf{I}, q > 0$.

We can further suppose that H.C.F. of $p$ and $q$ is 1.

Clearly $2 < \sqrt{8} < 3 \Rightarrow 2 < \frac{p}{q} < 3 \Rightarrow 2q < p < 3q \Rightarrow 0 < p - 2q < q$

$\Rightarrow$ $p - 2q$ is a positive integer less than $q$.

$\therefore$ $\sqrt{8}\,(p - 2q)$ is not an integer. ...(1)

Again $\sqrt{8}\,(p-2q) = \frac{p}{q}(p-2q) = \frac{p^2}{q} - 2p = \frac{p^2}{q^2} \cdot q - 2p = 8q - 2p$,

where $8q - 2p$ is an integer.

Thus $\sqrt{8}\,(p - 2q)$ is an integer. ...(2)

(1) and (2) are contradictory statements. Hence $\sqrt{8}$ is not a rational number.

**Example 3.** *Prove that there is no rational number whose square is* 10. [D.U., B.C.A. 1998]

**Hint.** Let $\sqrt{10} = p/q$, where $p, q \in \mathbf{I}, q > 0, (p, q) = 1$. Take $3 < \sqrt{10} < 4$ and proceed like Example 2.

**Example 4.** *Prove that there is no rational number whose cube is* 2.

**Solution.** Let, if possible, $x^3 = 2$ ; $x \in \mathbf{Q}$.

We may take $x = p/q$, where $p$ and $q$ are co-prime integers *i.e.*, the H.C.F. of $p$ and $q$ is 1. We have $x^3 = 2 \Rightarrow p^3 = 2q^3 \Rightarrow p^3$ is a multiple of 2 and so $p$ must be a multiple of 2, for otherwise $p^3$ will not a multiple of 2. Let $p = 2r$, for some $r \in \mathbf{Z}$. Putting $p = 2r$ in $p^3 = 2q^3$, we get $q^3 = 4r^3 \Rightarrow q^3$ is a multiple of 2 and so $q$ must be a multiple of 2. Sine $p$ and $q$ are both multiples of 2, it follows that 2 divides $p$ and $q$, which contradicts the fact that $p$ and $q$ are co-prime. Hence our assumption is wrong and so there is no rational number whose cube is 2.

**Example 5.** *Prove that there is no rational number whose square is* 12.

**Solution.** Let, if possible, $\sqrt{12} = p/q$, where $p, q \in \mathbf{Z}$ and $q \neq 0$. Then $2\sqrt{3} = p/q \Rightarrow \sqrt{3} = p/2q \Rightarrow \sqrt{3}$ is a rational number, which is a contradiction. Hence $\sqrt{12}$ is an irrational number.

## EXERCISES

1. Show that there is no rational number whose square is (*i*) 1 (*ii*) 5, (*iii*) 6.
2. Prove that there is no rational number whose square is 18.
3. If $p$ is any prime number, show that $\sqrt{p}$ is not a rational number.

   **Hint.** Let $\sqrt{p} = m/n$, where $m$ and $n$ are co-prime integers. Then $m^2 = pn^2 \Rightarrow m^2$ is a multiple of $p \Rightarrow m$ is a multiple of $p$, since $p$ is prime. Let $m = rp$ for some $r \in \mathbf{Z}$. Then $m^2 = pn^2 \Rightarrow n^2 = pr^2 \Rightarrow n^2$ is a multiple of $p \Rightarrow n$ is a multiple of $p$ Thus $p$ divides both $m$ and $n$, a contradiction.

Notice that $0 \notin S$ and $1 \in S$.

**6.** The set $T = \left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, \ldots, -\frac{n+1}{n}, \ldots\right\}$ is bounded, since $-2 \leq x < -1 \ \forall \ x \in T$.

Notice that $-2 \in T$ and $-1 \notin T$

**Remarks.** From the above illustrations, the following points are clear

**1.** *A set may or may not possess upper and/or lower bounds.*

*e.g.*, **N** has a lower bound but has no upper bound.

**Z** has neither an upper bound nor a lower bound.

**2.** *Bounds of a set may or may not belong to the set.*

If $S = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$, its lower bound $0 \notin S$ and upper bound $1 \in S$.

If $T = \left\{-\frac{n+1}{n} : n \in \mathbf{N}\right\}$, its lower bound $-2 \in T$ and upper bound $-1 \notin T$.

**3.** *If a real number $K$* **is not an upper bound** *of a set $S$, then there exists* **at least** *one element $y \in S$ such that $y > K$.*

**4.** *If a real number $k$* **is not a lower bound** *of a set $S$, then there exists* **at least** *one element $z \in S$ such that $k > z$ i.e., $z < k$.*

**5.** *If a set has one upper bound, then it has an infinite number of upper bounds.*

Notice that if $K$ is an upper bound of a set $S$, then any number $K_1 (> K)$ is also an upper bound of $S$.

**6.** *If a set has one lower bound, then it has an infinite number of lower bounds.*

Notice that if $k$ is a lower bound of a set $S$, then any number $k_1 (< k)$ is also a lower bound of $S$.

Remarks 5 and 6 lead to the following :

**Definition 4.** *The smallest of all the upper bounds of a set $S$ is called the* **supremum** *of the set $S$ or the* **least upper bound** *(l.u.b.) of the set $S$.*

**Definition 5.** *The greatest of all the lower bounds of a set $S$ is called the* **infimum** *of the set $S$ or the* **greatest lower bound** *(g.l.b.) of the set $S$.*

**Notation.** If $M$ and $m$ are the supremum and infimum of a set $S$, respectively, we write

$$m = \inf S \quad \text{and} \quad M = \sup S.$$

$m$ and $M$ are called the **bounds** of the set $S$, provided they exist.

**Remarks.**

**7.** *A set may or may not have the supremum or/and infimum.*

**8.** *The supremum and infimum of a set may or may not belong to the set.*

**Illustrations**

7. If $S = \{1, 2, 3, \dots, 10\}$, then $\inf S = 1 \in S$, $\sup S = 10 \in S$.

8. If $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$, then $\inf S = 0 \notin S$, $\sup S = 1 \in S$.

9. If $S = \left\{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\right\}$, then $\inf S = -1 \in S$, $\sup S = 0 \notin S$.

10. If $S = \left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, \dots, -\frac{n+1}{n}, \dots\right\}$, then $\inf S = -2 \in S$ and $\sup S = -1 \notin S$.

11. If $S = \left\{1-1, 1+\frac{1}{2}, 1-\frac{1}{3}, 1+\frac{1}{4}, 1-\frac{1}{5}, 1+\frac{1}{6}, \dots, 1+\frac{(-1)^n}{n}, \dots\right\}$

$= \left\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}, \dots\right\}$, then $\inf S = 0 \in S$, $\sup S = \frac{3}{2} \in S$.

12. The set **Z** of integers has neither infimum nor supremum.

## EXAMPLES

✓ **Example 6.** *Give an example of a set S which is :*

✓ *(i) bounded above but not bounded below.*

*(ii) bounded below but not bounded above.*

*(iii) neither bounded above nor bounded below.* ✓*(iv) bounded.*

**Solution.** *(i)* $S = \{\dots, -3, -2, -1\}$

is bounded above by $-1$, but not bounded below.

*(ii)* $\mathbf{N} = \{1, 2, 3, \dots\}$

is bounded below by 1, but not bounded above.

*(iii)* $\mathbf{I} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

is neither bounded above nor bounded below.

*(iv)* $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\}$

is bounded, since $0 < \frac{1}{n} \le 1 \ \forall\, n \in \mathbf{N}$.

✓ **Example 7.** *Give an example of a non-empty bounded subset S of* **R** *whose supremum and infimum both belong to* $\mathbf{R} - S$.

**Solution.** Let $S = \{x : 0 < x < 1\}$. Then $\sup S = 1 \in \mathbf{R} - S$ and $\inf S = 0 \in \mathbf{R} - S$. over here $x < 1$ supremum lies outside

✓ **Example 8.** *Which of the following sets are bounded above, bounded below, bounded or otherwise :*

*(i)* $A = \{3, 3^2, 3^3, \dots, 3^n, \dots\}$, *(ii)* $B = \left\{2, \frac{3}{2}, \frac{4}{3}, \dots, \frac{n+1}{n}, \dots\right\}$,

*(iii)* $C = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$, *(iv)* $D = \{-1, 2, -3, 4, \dots, (-1)^n n, \dots\}$,

$\left\{1 + \frac{(-1)^n}{n} : n \in \mathbf{N}\right\}$, *(vi)* $F = \left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots\right\}$,

---

*(vii)* $G = \left\{-2, -\frac{3}{2}, -\frac{4}{3}, \dots, -\frac{n+1}{n}, \dots\right\}$,

*(viii)* $S = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots (-1)^n \cdot \frac{1}{n}, \dots\right\}$, imp. [D.U., Maths (H) 1999, 97]

*(ix)* $S = \left\{\frac{4n+3}{n} : n \in \mathbf{N}\right\}$, imp.

*(x)* $S = \left\{\sin\frac{\pi}{6}, \sin\frac{2\pi}{6}, \sin\frac{3\pi}{6}, \dots, \sin\frac{n\pi}{6}, \dots\right\}$. imp.

*Find the supremum and infimum of the above sets, if they exist.*

**Solution.** *(i)* A is bounded below by 3, but is not bounded above. $\inf A = 3 \in A$ but $\sup A$ does not exist.

*(ii)* $B$ is bounded as $1 < x \le 2 \ \forall\, x \in B$.

$\inf B = 1 \notin B$, $\sup B = 2 \in B$.

*(iii)* $C = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$.

$C$ is bounded, as $0 < x \le 1 \ \forall\, x \in C$.

$\inf C = 0 \notin C$, $\sup C = 1 \in C$.

*(iv)* $D$ is neither bounded above nor bounded below.

*(v)* $E = \left\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}, \dots\right\}$.

$E$ is bounded, as $0 \le x \le \frac{3}{2} \ \forall\, x \in E$.

$\inf E = 0 \in E$, $\sup E = \frac{3}{2} \in E$.

over here $\frac{3}{2}$ is supremum because it is least upper bound as compared to other bounds after $\frac{3}{2}$

*(vi)* $F = \left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots\right\}$.

$F$ is bounded, since $-1 \le x \le -\frac{1}{2} \ \forall\, x \in F$.

$\inf F = -1 \in F$, $\sup F = -\frac{1}{2} \in F$.

*(vii)* $G = \left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, \dots\right\}$.

$G$ is bounded, since $-2 \le x \le -1 \ \forall\, x \in G$.

$\inf G = -2 \in G$, $\sup G = -1 \notin G$.

*(viii)* $S = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \dots\right\}$ is bounded.

$\inf S = -1 \in S$, $\sup S = \frac{1}{2} \in S$.

*(ix)* $S = \left\{7, 5\frac{1}{2}, 5, 4\frac{3}{4}, 4\frac{3}{5}, 4\frac{3}{6}, \dots\right\}$ is bounded.

$\sup S = 7 \in S$, $\inf S = 4 \notin S$.

(x) $T = \left\{ \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, \ldots \right\}$ is bounded.

$\sup T = 1 \in T, \qquad \inf T = -1 \in T.$

**Example 9.** *Find the supremum and infimum of the set :*

$$S = \left\{ 1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2}, \ldots, 1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}, \ldots \right\}.$$

**Solution.** We have

$$1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}$$

It is a G.P

$S_n = a\left(\frac{1-r^n}{1-r}\right)$

$$\frac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right)$$

$$= 2 - \frac{1}{2^{n-1}} < 2 \ \forall \ n \in \mathbf{N}.$$

Hence $\sup S = 2, \quad \inf S = 1.$

**Example 10.** *The supremum of a non-empty set S, if it exists, is always unique.* [D.U., Maths. (H) 1995, 94]

**Solution.** Let, if possible, $k, k'$ be two suprema of $S$.

Now $k' = \sup S$ and $k$ is an upper bound of $S \Rightarrow k' \leq k$. ...(*i*)

Again $k = \sup S$ and $k'$ is an upper bound of $S \Rightarrow k \leq k'$. ...(*ii*)

From (*i*) and (*ii*), $k = k'$.

Hence the set $S$ has the unique supremum.

**Example 11.** *Prove that inf S, if it exists, is always unique.*

**Solution.** Please try yourself.

**Example 12.** *Prove that the greatest member of a set, if it exists, is the supremum (l.u.b.) of the set.*

**Solution.** *An element G of a set S is called the* **greatest member** *of S if (i) $G \in S$ and (ii) $x \leq G \ \forall \ x \in S$.*

Let $G$ be the greatest member of $S$.

Then $x \leq G \ \forall \ x \in S$ and so $G$ is an upper bound of $S$.

Let $G_1$ be any number $< G$, $G \in S$. Then $G_1$ cannot be an upper bound of $S$. ($\because G_1 < G$ and $G \in S$). Hence $G$ is the l.u.b. (supremum) of $S$.

**Example 13.** *Show that the smallest member of a set, if it exists, is the infimum (g.l.b.) of the set.*

**Solution.** *An element g of a set S is called the* **smallest member** *of S if (i) $g \in S$ and (ii) $x \geq g \ \forall \ x \in S$.*

Let $s$ be the smallest member of a set $S$.

Then $s \in S$ and $s \leq x \ \forall \ x \in S \Rightarrow s$ is a lower bound of $S$.

Let $s_1$ be any number $> s$.

Then $s_1 > s$ and $s \in S \Rightarrow s_1$ cannot be a lower bound of $S$.

Hence $s$ is the g.l.b. (infimum) of the set $S$.

**Example 14.** *Define the notions supremum and infimum and distinguish them from the greatest and the least points. Give an example of a set of irrationals that has a rational supremum.* [B.C.A. Sample Problem]

**Solution.** The infimum and supremum of a set may or may not belong to the set but the greatest and least points of a set must belong to the set. Indeed if the greatest and least points of a set $S$ exist, then there are the supremum and infimum of $S$, respectively. [See Examples 12 and 13]

The set $\quad S = \{\sqrt{n+1} - \sqrt{n} : n \in \mathbf{N}\}$

*i.e.*, $\quad S = \{\sqrt{2} - \sqrt{1}, \sqrt{3} - \sqrt{2}, \sqrt{4} - \sqrt{3}, \ldots\}$

is a set of irrational members having $1 \in \mathbf{Q}$ as the supremum.

Notice that $\sqrt{n+1} - \sqrt{n} < 1 \ \forall \ n \in \mathbf{N}$, since

$$(1+\sqrt{n})^2 - (\sqrt{n+1})^2 = 2\sqrt{n} > 0 \quad \forall \quad n \in \mathbf{N}$$

$$\Rightarrow 1+\sqrt{n} > \sqrt{n+1} \ \Rightarrow \ 1 > \sqrt{n+1} - \sqrt{n} \quad \forall \quad n \in \mathbf{N}.$$

**Example 15.** *If S is a bounded non-empty subset of* **R**, *then prove that : inf S ≤ sup S.* [D.U., Maths (H) 1999]

**Solution.** Since $S$ is non-empty, we can choose some $x \in S$. By definition, $\inf S \leq x$ and $x \leq \sup S$. Hence $\inf S \leq \sup S$.

**Example 16.** *If $S \subseteq T \subseteq$* **R**, *where $S \neq \phi$, then show that*

(*i*) *If T is bounded above, then sup S ≤ sup T;*

(*ii*) *If T is bounded below, then inf T ≤ inf S.*

**Solution.** (*i*) Let $\alpha = \sup T$. By definition, $\alpha \geq y \ \forall \ y \in T$.

Since $\quad S \subseteq T, \alpha \geq x \quad \forall \ x \in S$.

It follows that $\alpha$ is an upper bound of $S$ and so $\sup S \leq \alpha$.

Hence $\quad \sup S \leq \sup T$.

(*ii*) Let $\beta = \inf T$. By definition, $\beta \leq y \ \forall \ y \in T$.

Since $\quad S \subseteq T, \beta \leq x \quad \forall \quad x \in S$.

It follows that $\beta$ is a lower bound of $S$ and so $\beta \leq \inf S$.

Hence $\quad \inf T \leq \inf S$.

**Example 17.** *If S and T are non-empty bounded subsets of real numbers, then show that $S \subseteq T \Rightarrow \inf T \leq \inf S \leq \sup S \leq \sup T$.*

**Solution.** By Example 16(*ii*), $\inf T \leq \inf S$. ...(i)

By Example 15, $\inf S \leq \sup S$. ...(ii)

By Example 16(*i*), $\sup S \leq \sup T$. ...(iii)

The result follows by (i), (ii), (iii).

**Example 18.** *Let S and T be non-empty subsets of* **R** *with the property :*

$$s \leq t \quad \forall \quad s \in S, t \in T. \qquad \ldots(1)$$

*(i) Show that $S$ is bounded above and that $T$ is bounded below.*

*(ii) Show that* $\sup S \le \inf T$. [D.U., Maths (H) 1999, 1997].

**Solution.** Let $t \in T$ be arbitrary. Then $s \le t \ \forall \ s \in S$, by (1). It follows that $S$ is bounded above.

Let $s \in S$ be arbitrary. Then $s \le x \ \forall \ x \in T$, by (1). Thus $s$ is a lower bound of $T$ and so $T$ is bounded below.

Also $\quad s \le \inf T \quad \forall \quad s \in S \qquad (\because \ \inf T = g.l.b.\ T)$

$\Rightarrow \inf T$ is an upper bound of $S$.

$\Rightarrow \sup S \le \inf T. \qquad (\because \ \sup S = l.u.b.\ S)$

**Example 19.** *Let $A$ and $B$ be two non-empty subsets of* **R** *and let*

$$C = \{x + y : x \in A \ \text{ and } \ y \in B\}.$$

*If each of $A$ and $B$ has a supremum, show that $C$ has a supremum and*

$$\sup C = \sup A + \sup B. \qquad \text{[D.U., B.Sc. (G) 1997]}$$

**Solution.** Let $a = \sup A$ and $b = \sup B$. ...(1)

We shall prove that $a + b = \sup C$.

Let $z \in C$ be arbitrary. Then $z = x + y$, for some $x \in A$ and $y \in B$.

We have $\quad x \le a$ and $y \le b$, by (1)

$\Rightarrow \qquad x + y \le a + b \Rightarrow z \le a + b \ \forall \ z \in C.$

Thus $a + b$ is an upper bound of $C$. ...(2)

**Let $\alpha$ be any real number such that $\alpha < a + b$.** ...(3)

Then $\alpha - b < a$ and $a = \sup A$, therefore $\alpha - b$ is not an upper bound of $A$. Consequently, there exists some $x \in A$ such that

$$\alpha - b < x \quad \text{or} \quad \alpha - x < b = \text{l.u.b. } B$$

$\Rightarrow \alpha - x$ is not an upper bound of $B$ and so there exists some $y \in B$ such that

$$\alpha - x < y \quad \text{or} \quad \alpha < x + y, \text{ where } x + y \in C$$

$\Rightarrow \alpha$ is not an upper bound of $C$. ...(4)

From (2), (3) and (4), it follows that $a + b$ is the least upper bound of $C$.

Hence $\quad \sup C = a + b = \sup A + \sup B.$

**Example 20.** *Let $A$ and $B$ be two non-empty subsets of* **R** *and let*

$$C = \{x + y : x \in A \text{ and } y \in B\}.$$

*If each of $A$ and $B$ has an infimum, show that $C$ has an infimum and*

$$\inf C = \inf A + \inf B.$$

**Solution.** Let $\quad a = \inf A$ and $b = \inf B$. ...(1)

We shall prove that $a + b = \inf C$.

Let $z \in C$ be arbitrary so that $z = x + y$, for some $x \in A$ and $y \in B$.

We have $\quad a \le x$ and $b \le y$, by (1)

$\Rightarrow \qquad a + b \le x + y \ \Rightarrow a + b \le z, \forall \ z \in C.$

[illegible]s $a + b$ is a lower bound of $C$. ...(2)

[illegible] **$\alpha$ be any real number such that $\alpha > a + b$.** ...(3)

Then $\qquad \alpha - b > a = \inf A$

$\Rightarrow \alpha - b$ is not a lower bound of $A \qquad (\because \ a = g.l.b.\ A)$

$\Rightarrow$ there exists some $x \in A$ such that $\alpha - b > x$

$\Rightarrow \alpha - x > b = \inf B$

$\Rightarrow \alpha - x$ is not a lower bound of $B \qquad (\because \ b = g.l.b.\ B)$

$\Rightarrow$ there exists some $y \in B$ such that $\alpha - x > y$

$\Rightarrow \alpha > x + y$, where $x + y \in C$ ...(4)

$\Rightarrow \alpha$ is not a lower bound of $C$.

From (2), (3) and (4), it follows that $a + b$ is the greatest lower bound of $C$.

Hence $\inf C = a + b = \inf A + \inf B$.

**Example 21.** *Let $A$ and $B$ be two non-empty bounded sets of* **positive** *numbers and let*

$$C = \{xy : x \in A \text{ and } y \in B\}.$$

*Show that $C$ is bounded and that (i) $\sup C = \sup A \,.\, \sup B$, (ii) $\inf C = \inf A \,.\, \inf B$.*

**Solution.** Let $\quad a = \sup A, \quad b = \sup B$. ...(1)

Clearly, $\quad a > 0$ and $b > 0$.

Let $z \in C$ be arbitrary so that $z = xy$, for some $x \in A$ and $y \in B$.

We have $\quad x \le a$ and $y \le b$, by (1)

$\Rightarrow xy \le ab$, as $x, y, a, b$ are all positive

$\Rightarrow \quad z \le ab \ \forall \ z \in C$ ...(2)

Thus $ab$ is an upper bound of $C$. ...(3)

**Let $\alpha$ be any real number such that $\alpha < ab$.**

Then $\dfrac{\alpha}{b} < a \qquad (\because \ b > 0) \quad$ and $\quad a = \sup A$

$\Rightarrow \alpha/b$ is not an upper bound of $A$

$\Rightarrow$ there exists some $x \in A$ such that $\alpha/b < x$

$\Rightarrow \alpha/x < b \qquad (\because \ x > 0) \quad$ and $\quad b = \sup B$

$\Rightarrow \alpha/x$ is not an upper bound of $B$

$\Rightarrow$ there exists some $y \in B$ such that $\alpha/x < y$

$\Rightarrow \alpha < xy \qquad (\because \ x > 0), \quad xy \in C$ ...(4)

$\Rightarrow \alpha$ is not an upper bound of $C$.

From (2), (3) and (4), it follows that

$$\sup C = ab = \sup A \,.\, \sup B.$$

The proof of part (*ii*) is on the similar lines.

**Example 22.** *If $S$ and $T$ are arbitrary non-empty bounded subsets of* **R**, *then show that*

$$\sup(S \cup T) = \max(\sup S, \sup T). \qquad \text{[D.U., Maths (H) 1999]}$$

...(1)

**Solution.** Let $\quad s = \sup S, \quad t = \sup T$.

Let $x \in S \cup T$ be arbitrary. Then $x \in S$ or $x \in T$

$\Rightarrow x \le s$ or $x \le t$, using (1)

$\Rightarrow x \le \max(s, t)\ \forall\ x \in S \cup T$.

Thus $\max(s, t)$ is an upper bound of $S \cup T$. ...(2)

**Let $\alpha$ be any real number such that $\alpha < \max(s, t)$.** ...(3)

$\Rightarrow \alpha < s$ or $\alpha < t$.

If $\alpha < s$, then $\alpha$ is not an upper bound of $S$ $\quad (\because\ s = l.u.b.\ S)$

$\Rightarrow$ there exists some $y \in S$ (and hence $y \in S \cup T$) such that $y > \alpha$

$\Rightarrow \alpha$ is not an upper bound of $S \cup T$. ...(4)

From (2), (3) and (4), it follows that

$$\sup(S \cup T) = \max(s, t) = \max(\sup S, \sup T).$$

Similarly, we can discuss the case if $\alpha < t$.

**Example 23.** *Let A be a non-empty set of positive numbers, which is bounded above and let*

$$B = \{x \in \mathbf{R} : 1/x \in A\}.$$

*Prove that B is bounded below and* $\inf B = \dfrac{1}{\sup A}$.

**Solution.** Let $a = \sup A$. Then $a > 0$.

We shall prove that $\inf B = 1/a$.

Let $x \in B$ be arbitrary so that $1/x \in A$ and $x > 0$

$\Rightarrow 1/x \le a \qquad (\because\ a = \sup A)$

$\Rightarrow x \ge 1/a\ \forall\ x \in B$

$\Rightarrow 1/a$ is a lower bound of $B$. ...(1)

Hence $B$ is bounded below.

**Let $\alpha$ be any positive real number such that $\alpha > 1/a$.** ...(2)

$\Rightarrow 1/\alpha < a \Rightarrow 1/\alpha$ is not an upper bound of $A$ $\quad (\because\ a = l.u.b.\ A)$

$\Rightarrow$ there exists some $x \in A$ (*i.e.*, $x > 0$) such that

$1/\alpha < x$ or $\alpha > 1/x$, where $1/x \in B$

$\Rightarrow \alpha$ is not a lower bound of $B$. ...(3)

From (1), (2) and (3), it follows that

$$\frac{1}{a} = \inf B.$$

Hence $\qquad \inf B = \dfrac{1}{a} = \dfrac{1}{\sup A}$.

**Example 24.** *Let A be a non-empty subset of* **R** *having a supremum,* [illegible] $B = \{2x : x \in A\}$. *Show that* $\sup B = 2 \sup A$.

[illegible] Let $a = \sup A$. We shall prove that $\sup B = 2a$.

[illegible] be arbitrary. Then $y = 2x$, for some $x \in A$

[illegible] $x \le a \qquad (\because\ a = \sup A)$

[illegible] $x \le 2a \Rightarrow y \le 2a\ \forall\ y \in B$.

[illegible] upper bound of $B$. ...(1)

**Let $\alpha$ be any real number such that $\alpha < 2a$.** ...(2)

$\Rightarrow \alpha/2 < a \Rightarrow \alpha/2$ is not an upper bound of $A$ $\quad (\because\ a = l.u.b.\ A)$

$\Rightarrow$ there exists some $x \in A$ such that $x > \alpha/2 \Rightarrow 2x > \alpha$, where $2x \in B$

$\Rightarrow \alpha$ is not an upper bound of $B$. ...(3)

From (1), (2) and (3), it follows that $2a$ is least upper bound of $B$. Hence

$$\sup B = 2a = 2 \sup A.$$

**Example 25.** *Let A be a non-empty subset of* **R** *having a supremum.*

*Let* $\quad B = \{x \in \mathbf{R} : x - 1 \in A\},\ C = \left\{x \in \mathbf{R} : \dfrac{x+1}{2} \in A\right\}$.

*Prove that (i)* $\sup B = 1 + \sup A$, *(ii)* $\sup C = 2 \sup A - 1$.

**Please try yourself.**

**Example 26.** *Let A and B be two non-empty bounded subsets of* **R** *and let*

$$C = \{x - y : x \in A, y \in B\}.$$

*Prove that* $\sup C = \sup A - \inf B$.

**Hint.** Similar to Examples 19 and 20.

**Example 27.** *Show that :* $\sup\{r \in \mathbf{Q} : r < a\} = a$, *for each* $a \in \mathbf{R}$.

[D.U., Maths (H) 1999]

**Solution.** Let $S = \{r \in \mathbf{Q} : r < a\}$. Let $x \in S$ be arbitrary.

Then $x < a\ \forall\ x \in S \Rightarrow a$ is an upper bound of $S$. ...(1)

Let $\alpha$ be any real number such that $\alpha < a$. ...(2)

Since $\alpha$ and $a$ are real numbers and $\alpha < a$, there exists a rational number $q$ such that $\alpha < q < a$. Now

$$q \in \mathbf{Q} \text{ and } q < a \Rightarrow q \in S, \text{ where } \alpha < q$$

$\Rightarrow \alpha$ is not an upper bound of $S$. ...(3)

From (1), (2) and (3), it follows that $\quad a = l.u.b.\ S = \sup S$.

**Example 28.** *Give two examples of sets S such that sup S = 2 and sup S < 2.*

**Solution.** Refer to Example 27. Let $S = \{x \in \mathbf{Q} : x < 2\}$. Then $\sup S = 2$.

Let $S = \{x \in \mathbf{Q} : x < 1\frac{1}{2}\}$. Then $\sup S = 1\frac{1}{2} < 2$.

The students are advised to construct other examples without using the result of Example 27.

**Example 29.** *Let A be a non-empty set of real numbers which has a supremum and let* $B = \{x : -x \in A\}$. *Show that B is bounded below and*

$$\inf B = -\sup A.$$

**Solution.** Let $a = \sup A$. Let $x \in B$ be a arbitrary so that $-x \in A$.

We have $-x \le a \Rightarrow x \ge -a\ \forall\ x \in B \Rightarrow -a$ is a lower bound of $B$ $\Rightarrow B$ is bounded below. We now show that $-a = \inf B$. Let $\beta$ be *any* lower bound of $B$.

**A.M.** $a(b+c) = ab + ac \ \forall \ a, b, c \in \mathbf{R}$. *(Distributive law)*

**O.1.** For any $a, b \in \mathbf{R}$, exactly one of the following is true : *(Trichotomy law)*
(*i*) $a > b$ (*ii*) $a = b$ (*iii*) $b > a$.

**O.2.** $a > b$ and $b > c \Rightarrow a > c \ \forall \ a, b, c \in \mathbf{R}$. *(Transitive law)*

**O.3.** $a > b \Rightarrow a + c > b + c \ \forall \ a, b, c \in \mathbf{R}$.

**O.4.** $a > b$ and $c > 0 \Rightarrow ac > bc$.

The **order-completeness (O.C.)** property of **R** states that :

**(O.C.) Every non-empty set of real numbers which is bounded above, has the supremum in R.**

On account of **R** satisfying the above 16 properties, we say that

**The set R of real numbers is a complete ordered field.**

**Remarks. 1.** On account of **R** satisfying the first 11 properties *viz.* **A.1. – A.5., M.1. – M.5., A.M.** ; we say that **R is a field**. Since **R** also satisfies 4 *order properties viz.* **O.1. – O.4.**, we say that **R is an ordered field.**

**2.** It is easy to verify that the set **Q** of all rational numbers is also an ordered field *i.e.*, **Q** satisfies the 15 properties *viz.* **A.1. – A.5., M.1. – M.5., A.M., O.1. – O.4.** However, **Q** does not satisfy the **O.C.** property as shown below :

**Theorem 1.** *The set of rational numbers is not order complete.*
[D.U., B.Sc. (G) 1997, 94 ; B.C.A. 1999]

**Proof.** We shall give an example of a non-empty bounded above subset of **Q** which has no supremum in **Q**.

Let $S = \{x : x \in \mathbf{Q}, x > 0 \text{ and } x^2 < 2\}$.

Then $S \neq \phi$ ($\because$ $1 \in S$) and 2 is an upper bound of $S$.

Thus $S$ is a non-empty bounded above subset of **Q**.

Let, if possible, $S$ have the supremum, say $p \in \mathbf{Q}$,

*i.e.*, $p = \sup S$. Obviously $p > 0$.

By Trichotomy law, we have

$$p^2 < 2 \quad \text{or} \quad p^2 = 2 \quad \text{or} \quad p^2 > 2.$$

Now $p^2 = 2 \Rightarrow p = \sqrt{2} \in \mathbf{Q}$, which is impossible.

$\therefore$ $p^2 \neq 2$.

**Case I.** Let $p^2 < 2$.

Let $y = \dfrac{4+3p}{3+2p}$.

Then $y > 0$ and $y \in \mathbf{Q}$. ($\because$ $p \in \mathbf{Q}$ and $p > 0$)

Now $y^2 - 2 = \dfrac{p^2 - 2}{(3+2p)^2}$. ...(1)

$\therefore$ $p^2 < 2 \Rightarrow y^2 - 2 < 0 \Rightarrow y^2 < 2$ and $y > 0 \Rightarrow y \in S$.

**Ex. 4.** *Let S be a non-empty set of real numbers which is bounded below and let*

$$-S = \{-x \in \mathbf{R} : x \in S\}$$

*Prove that inf* $S = -\sup(-S)$. [B.C.A. Sample Problem]

**Solution.** Refer to Theorem 2. We have $T = -S$.

Let $K = \sup T = \sup(-S)$. ...(1)

As proved in Theorem 2, $-K = \inf S$. ...(2)

From (1) and (2), we have

$$\inf S = -K = -\sup(-S).$$

**Theorem 3.** *If S is a non-empty set of real numbers which is bounded above, then a real number s is the supremum of S if and only if the following two conditions hold :*

(*i*) $x \leq s \ \forall \ x \in S$.

(*ii*) *Given any* $\varepsilon > 0$, $\exists$ *some* $x \in S$ *such that* $x > s - \varepsilon$.

[D.U., B.A. (P) 1998, B.Sc. (G) 1999]

**Proof.** *The condition is necessary*. Let $s = \sup S = l.u.b.\ S$. By definition, $x \leq s$, $\forall \ x \in S$. This proves condition (*i*).

Let $\varepsilon > 0$. Then $s - \varepsilon < s$ cannot be an upper bound of $S$.

So there exists some $x \in S$ such that $x > s - \varepsilon$. This proves condition (*ii*).

*The condition is sufficient*. Let $s$ be any real number satisfying conditions (*i*) and (*ii*). We shall prove that $s = \sup S$.

Now $x \leq s \quad \forall \ x \in S \Rightarrow s$ is an upper bound of $S$.

Let, if possible, $s'$ by any real number $< s \Rightarrow s - s' > 0$.

Let $\qquad \varepsilon = s - s' > 0$.

By condition (*ii*), $\exists$ some $x \in S$ such that $x > s - \varepsilon = s - (s - s') = s'$

$\Rightarrow x > s'$ for some $x \in S$.

$\Rightarrow s'$ is not an upper bound of $S$.

We see that no real number $< s$ can be an upper bound of $S$.

Hence $\quad s = \sup S$.

**Theorem 4.** *If S is a non-empty subset of* **R** *which is bounded below, then a real number t is the infimum of S iff the following two conditions hold :*

(*i*) $x \geq t \ \forall \ x \in S$.

(*ii*) *Given any* $\varepsilon > 0$, $\exists$ *some* $x \in S$ *such that* $x < t + \varepsilon$.

[D.U., B.Sc. (G) 1998, 96, 93 ; Maths (H) 1998, 95]

**Proof.** *The condition is necessary*.

Let $\qquad t = \inf S = g.l.b.\ S$.

$\Rightarrow x \geq t \quad \forall \ x \in S$, which proves condition (*i*).

Let $\varepsilon > 0$. Then $t + \varepsilon > t$ cannot be a lower bound of $S$.

**Corollary 4.** *For any real number x, there exists an integer m such that* $m < x$.

**Proof.** Since $-x \in \mathbf{R}$, by Corollary 3, there exists a positive integer $n$ such that $n > -x \Rightarrow -n < x \Rightarrow m < x$, where $m = -n$.

## EXAMPLES

**Example 31.** *If x be any positive real number, then there exists a positive integer n such that* $\frac{1}{n} < x$.

**Solution.** Applying the **Archimedean** property, for $y = x > 0$ and $x = 1$, there exists a positive **integer** $n$ such that $ny > x \Rightarrow nx > 1 \Rightarrow n > 1/x$. Hence $1/n < x$.

**Example 32.** *State Archimedean property of real numbers. Hence show that for each real number* $\varepsilon > 0$, *there exists a positive integer m such that*

$$\frac{1}{n} < \varepsilon \text{ for each } n \geq m. \qquad \text{[B.C.A. Sample Problem]}$$

**Solution.** For $y = \varepsilon > 0$, $x = 1$ ; there exists a positive integer $m$ such that

$$my > x \Rightarrow m\varepsilon > 1 \Rightarrow m > \frac{1}{\varepsilon} \quad \Rightarrow \quad \frac{1}{m} < \varepsilon.$$

For each $n \geq m$, $\quad \frac{1}{n} \leq \frac{1}{m} < \varepsilon$.

Hence $\quad \frac{1}{n} < \varepsilon \quad \forall \quad n \geq m$.

**Example 33.** *If a and b are any two real numbers such that* $a \leq b + \frac{1}{n}$ *for all* $n \in \mathbf{N}$, *then show that* $a \leq b$.

**Solution.** Let, if possible, $a > b$. Then $a - b > 0$.

By the Archimedean property of real numbers, for $x = 1$ and $y = a - b > 0$, there exists a positive integer $m$ such that $m(a - b) > 1$ or $a > b + \frac{1}{m}$, which contradicts the given hypothesis. Hence $a \leq b$.

**Example 34.** *Prove that for any real number x, there exists a unique integer n such that* $n < x < n + 1$.

**Solution.** Let $S = \{t \in \mathbf{Z} : t \leq x\}$.

Then $S$ is a non-empty subset of **R** which is bounded above (by $x$). By order-completeness property of **R**, $S$ has the *l.u.b.* Let $n = l.u.b.\ S$. Since $x$ is an upper bound of $S$, $n < x$.

Now we prove that $x < n + 1$. Let, if possible, $x \geq n + 1 \Rightarrow n + 1 \leq x$, where $n + 1 \in \mathbf{Z} \Rightarrow n + 1 \in S \Rightarrow n + 1 \leq n$, since $n$ is an upper bound of $S$. This is impossible. Hence $n < x < n + 1$.

**Remark.** *The integer n satisfying* $n < x < n + 1$ *is called an* **integral part** *of x and is denoted by* $[x]$.

*that there exists an irrational number*

**Solution.** By Example 37, there exists a rational number $r$ such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}} \Rightarrow x < \sqrt{2}r < y \Rightarrow x < s < y,\ s = \sqrt{2}r.$$

We claim that $s$ is an irrational number, for if $s$ is a rational number, then $s/r\ (r \neq 0, r \in \mathbf{Q})$ is a rational number *i.e.*, $\sqrt{2}$ is a rational number, which is a contradiction.

## 1.9 Representation of Real Numbers as Points on a Straight Line

The points on a straight line can be used to represent real numbers. Let $X'X$ be any straight line. Mark two points $O$ and $A$ on it such that $A$ is to the right of $O$. The point $O$ divides the straight line into two parts. We call the part that lies to the right of $O$ as the *positive part*, and the part that lies to the left of $O$ the *negative part*. Such a line for which positive and negative sides are fixed is called a *directed line*.

Suppose the points $O$ and $A$ represent the numbers 0 and 1, respectively. With $OA$ as unity on a certain scale, we can locate all positive integers 1, 2, 3... on the right of $O$ and all negative integers ..., – 3, – 2, – 1 on the left of $O$. To represent any positive rational number $m/n$, we locate a point $P$ on the right of $O$ such that $OP$ is $m$ times the $n$th part of the unit length $OA$. Similarly, all negative rational numbers can be located on the left of $O$. The points on the straight line corresponding to rational numbers are called *rational points*. It might appear that rational points cover the entire straight line. But this is not true. There exist infinite number of points on the line which are not rational. For example, the point $Q$ on the line such that $OQ$ is equal to the diagonal of the square with side $OA$ $(OQ = \sqrt{2})$ does not correspond to any rational number. Such points on the line which are not rational indeed fill up the gaps between the rational points are called *irrational points* and these correspond to irrational numbers. In this way ***every real number can be represented as a point on the straight line***. On account of this representation, *we refer to a real number as a point of* **R** and the straight line is referred to as **real line.**

The following axiom is worth mentioning in this context :

### Dedekind-Cantor Axiom

*To every real number corresponds a unique point on a directed line and to every point on a directed line corresponds a unique real number.*

## 1.10 Intervals

The representation of real numbers as points on a straight line helps us to define four types of **intervals.**

Let $a$ and $b$ be any two real numbers such that $a < b$.

**1. Open interval**, *denoted by* $]a, b[$, *is defined as*

$$]a, b[\ = \{x : a < x < b\}.$$

The end points $a$ and $b$ do not belong to $]a, b[$.

**2. Closed interval**, *denoted by* $[a, b]$, *is defined as*

$$[a, b] = \{x : a \le x \le b\}.$$

The end points $a$ and $b$ belong to $[a, b]$.

**Semi-open or Semi-closed intervals** *are defined as follow* :

**3.** $]a, b] = \{x : a < x \le b\}$.

Here $a \notin\ ]a, b]$ and $b \in\ ]a, b]$.

The interval $]a, b]$ is said to be *open from the left and closed from the right.*

**4.** $[a, b[\ = \{x : a \le x < b\}$.

Here $a \in [a, b[$ and $b \notin [a, b[$.

The interval $[a, b[$ is said to be *open from the right and closed from the left.* The intervals $]a, b]$ and $[a, b[$ are also known as *half-open* or *half-closed* intervals.

**Remark.** In the next chapter we shall mainly consider open intervals of the form

$$]p - \varepsilon, p + \varepsilon[, \varepsilon > 0 \quad \text{and} \quad p \in \mathbf{R}.$$

Such an open interval *around* $p$ will be called a *neighbourhood* of the point $p$.

### 1.11 Absolute Value or Modulus of a Real Number

**Definition.** *Let a be any real number. The* **absolute value** *or* **modulus** *of a, denoted by* $|a|$, *is defined as*

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0. \end{cases}$$

It may be noted that $|a| \ge 0\ \forall\ a \in \mathbf{R}$.

**Remarks.** The following properties are true :

1. $|a| \ge 0\ \forall\ a \in \mathbf{R}$.
2. $|a| = |-a|$.

By the law of trichotomy, one and exactly one of the following is true
(i) $x > 0$ (ii) $x = 0$ (iii) $x < 0$

**Theorem 6.** *If a is any real number, then* $|a| = \max\{a, -a\}$.

**Proof.** We have $|a| = a$, if $a \ge 0$, $a \ge -a$ ...(1)

and $|a| = -a > a$, if $a < 0$. $-a > a$ ...(2)

From (1) and (2), we get $|a| \ge a$.

Again $|a| = a \ge -a$, if $a \ge 0$, ...(3)

$|a| = -a$, if $a < 0$. ...(4)

rom (3) and (4), we get $|a| \ge -a$.

ence $|a|$ is the maximum of the two real numbers $a$ and $-a$ *i.e.*,

$$|a| = \max\{a, -a.\}$$

**heorem 7.** *If* $x, y \in \mathbf{R}$ ; *then*

) $|x|^2 = x^2 = |-x|^2$.

) $|xy| = |x| . |y|$.

(*iii*) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}, y \ne 0.$

(*iv*) $|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$.

[D.U., B.Sc. (G) 1995 ; Maths (H) 1995]

**Proof.** (*i*) if $x \ge 0$, then $|x| = x \Rightarrow |x|^2 = x^2$,

and if $x < 0$, then $|x| = -x \Rightarrow |x|^2 = (-x)^2 = x^2$.

Thus $|x|^2 = x^2$.

Similarly, we can prove that $|-x|^2 = x^2$.

(*ii*) Using (*i*), $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2 . |y|^2 = (|x| . |y|)^2$

$\therefore$ $|xy| = \pm(|x| . |y|)$.

Since $|x|$ is non-negative $\forall\ x \in \mathbf{R}$, therefore

$$|xy| = |x| . |y|.$$

(*iii*) Using (*i*), $\left|\dfrac{x}{y}\right|^2 = \left(\dfrac{x}{y}\right)^2 = \dfrac{x^2}{y^2} = \dfrac{|x|^2}{|y|^2} = \left(\dfrac{|x|}{|y|}\right)^2$

$$\therefore \quad \left|\frac{x}{y}\right| = \pm\frac{|x|}{|y|}.$$

Since $|x|$ is non-negative $\forall\ x \in \mathbf{R}$, therefore

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \ne 0.$$

(*iv*) $|x + y|^2 + |x - y|^2 = (x + y)^2 + (x - y)^2$, by part (*i*)

$$= 2(x^2 + y^2)$$

$$= 2(|x|^2 + |y|^2), \text{ by part } (i).$$

**Theorem 8.** *If x, y are any two real numbers, then*

(*i*) $|x + y| \le |x| + |y|$. (*Triangle Inequality*)

[D.U., Maths (H) 1998 ; B.Sc. (G) 1995]

(*ii*) $|x - y| \ge ||x| - |y||$. [D.U., Maths (H) 1992 ; B.Sc. (G) 1995]

(*iii*) $|x - y| \le |x| + |y|$.

**Proof.** (*i*) To show $|x + y| \le |x| + |y|$.

**Case I.** Let $x + y \ge 0 \Rightarrow |x + y| = x + y$, by definition

$\le |x| + |y|$ $(\because x \le |x|\ \forall\ x \in \mathbf{R})$

$\therefore$ $|x + y| \le |x| + |y|$.

**Case II.** Let $x + y < 0 \Rightarrow |x + y| = -(x + y) = (-x) + (-y)$.

Since $-x \le |x|\ \forall\ x \in \mathbf{R}$. $\therefore$ $|x + y| \le |x| + |y|$.

Hence $|x + y| \le |x| + |y|$.

**Example 41.** *Show that*

(*i*) $\sqrt{x^2+y^2} \leq |x|+|y|$.

(*ii*) $\sqrt{|x+y|} \leq \sqrt{|x|}+\sqrt{|y|}$. [D.U., Maths (H) 1998]

**Solution.** (*i*) We know $|x|^2 = x^2 \Rightarrow |x| = +\sqrt{x^2}$. Thus

$$|x|+|y| = \sqrt{x^2}+\sqrt{y^2} \geq \sqrt{x^2+y^2}.$$

Hence $\sqrt{x^2+y^2} \leq |x|+|y|$.

(*ii*) We know $|x+y| \leq |x|+|y|$.

$\therefore \quad \sqrt{|x+y|} \leq \sqrt{|x|+|y|} \leq \sqrt{|x|}+\sqrt{|y|}$.

**Example 42.** (*i*) *Show that* $|a+b| = |a|+|b|$ *if and only if* $ab \geq 0$.

[D.U. Maths (H) 1994]

(*ii*) *Show that* $|a+b| < |a|+|b|$ *if and only if* $ab < 0$.

**Solution.** (*i*) We know $|a+b|^2 = (a+b)^2 = a^2+b^2+2ab$

Since $|a|^2 = a^2$ and $|b|^2 = b^2$, so

$$|a+b|^2 = |a|^2+|b|^2+2|ab| \quad (\because ab \geq 0 \Leftrightarrow |ab| = ab)$$

$$= |a|^2+|b|^2+2|a||b| \quad (\because |ab| = |a||b|)$$

$\therefore \quad |a+b|^2 = (|a|+|b|)^2$ if and only if $ab \geq 0$.

Hence $|a+b| = |a|+|b|$ if and only if $ab \geq 0$,

since modulus of a real number is non-negative.

(*ii*) We know $|a+b|^2 = (a+b)^2 = a^2+b^2+2ab$

$$< a^2+b^2+2|ab| \text{ iff } ab < 0$$

$$= |a|^2+|b|^2+2|a||b| = (|a|+|b|)^2.$$

Hence $|a+b| < |a|+|b|$ iff $ab < 0$.

**Example 43.** *Show that for all real numbers x and y,*

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|}+\frac{|y|}{1+|y|}.$$ [D.U., Maths (H) 1996]

**Solution.** We know $|x+y| \leq |x|+|y|$

$\Rightarrow 1+|x+y| \leq 1+|x|+|y|$

$$\Rightarrow \frac{1}{1+|x+y|} \geq \frac{1}{1+|x|+|y|} \Rightarrow -\frac{1}{1+|x+y|} \leq -\frac{1}{1+|x|+|y|}$$

$$\Rightarrow 1-\frac{1}{1+|x+y|} \leq 1-\frac{1}{1+|x|+|y|} \Rightarrow \frac{|x+y|}{1+|x+y|} \leq \frac{|x|+|y|}{1+|x|+|y|}$$

$$\Rightarrow \frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|+|y|}+\frac{|y|}{1+|x|+|y|} \leq \frac{|x|}{1+|x|}+\frac{|y|}{1+|y|}$$

$(\because 1+|x|+|y| \geq 1+|x|$ and $1+|x|+|y| \geq 1+|y|)$

**Example 44.** *Show that for all real numbers x and y :*

$$\frac{|x-y|}{1+|x-y|} = \frac{|x|}{1+|x|}+\frac{|y|}{1+|y|}.$$ [D.U., B.C.A., 1999]

**Example 47.** *Examine if the following statements are true :*

(*i*) $x \le y \Rightarrow |x| \le |y|$ ?

(*ii*) $|x+y| \le 1 \Rightarrow |x| \le 1$ *or* $|y| \le 1$ ?

**Solution.** The above statements are not true, in general.

(*i*) Let $x = -1, y = \frac{1}{2}$. Then $x < y$, but $|x| > |y|$.

(*ii*) Let $x = -2$, $y = \frac{3}{2}$. Then $|x+y| = \frac{1}{2} < 1$, but $|x| > 1$ and $|y| > 1$.

**Example 48.** *Find a positive* $\delta$ *such that* $|x| < \delta \Rightarrow |x^2 + 2x| < 1$.

**Solution.** Let $|x| < \delta$. Then

$$|x^2 + 2x| = |x| . |x+2| \le |x| \{|x| + |2|\} < \delta(\delta + 2).$$

$\therefore$ $|x^2 + 2x| < \delta(1+2) = 3\delta$, if $\delta < 1$.

Hence $|x^2 + 2x| < 1$, if $\delta < \frac{1}{3}$.

Hence $|x| < \frac{1}{3} \Rightarrow |x^2 + 2x| < 1, \delta = \frac{1}{3}$.

**Example 49.** *Show that for any two real numbers* $a$ *and* $b$, [D.U., B.Sc. (G) 1988]

(*i*) *Max* $\{a, b\} = \frac{1}{2}\{a + b + |a-b|\}$.

(*ii*) *Min* $\{a, b\} = \frac{1}{2}\{a + b - |a-b|\}$.

**Solution.** (*i*) By Trichotomy law, one of the following must be true :

(*i*) $a > b$, (*ii*) $a = b$, (*iii*) $b > a$.

If $a = b$, then Max $\{a, a\} = a$,

and $\frac{1}{2}\{a + b + |a-b|\} = \frac{1}{2}(a + a + 0) = a$.

Thus Max $\{a, b\} = \frac{1}{2}\{a + b + |a - b|\}$. ...(1)

Let $a > b$ so that $|a-b| = a-b$. $(\because\ a - b > 0)$

$\therefore$ $\frac{1}{2}\{a + b + |a-b|\} = \frac{1}{2}(a + b + a - b) = a$.

Again Max $\{a, b\} = a$ $(\because\ a > b)$. Thus (1) is true.

Let $b > a$. Then $a < b \Rightarrow a - b < 0 \Rightarrow |a - b| = -(a-b)$.

$\therefore$ $\frac{1}{2}\{a + b + |a-b|\} = \frac{1}{2}(a + b - a + b) = b$.

Since $b > a$, so Max $\{a, b\} = b$. Thus (1) is true again.

Hence the result.

(*ii*) Now we show Min $\{a, b\} = \frac{1}{2}\{a + b - |a - b|\}$. ...(2)

Let $a = b$ so that Min $\{a, a\} = a$,

and $\frac{1}{2}\{a + b - |a - b|\} = \frac{1}{2}(a + a - 0) = a$.

Thus (2) is true.

Let $a > b$ so that Min $\{a, b\} = b$.

Now $a > b \Rightarrow a - b > 0 \Rightarrow |a - b| = a - b$.

$\therefore$ $\frac{1}{2}\{a + b - |a - b|\} = \frac{1}{2}(a + b - a + b) = b =$ **Min** $\{a, b\}$.

Let $b > a$ so that Min $\{a, b\} = a$.

# 2

# Neighbourhoods and Limit Points

In this chapter we shall discuss some basic concepts of Real Analysis such as *neighbourhood of a point, open and closed sets, limit point of a set* and prove some of their important properties. We recall that an open interval is of the form : $]a, b[ = \{x : a < x < b\}$. Here $a \notin\ ]a, b[$ and $b \notin\ ]a, b[$.

An open interval around a point $p \in \mathbf{R}$ is $]p - \varepsilon, p + \varepsilon[$, $\varepsilon > 0$.

## 2.1 Neighbourhood of a Point

**Definition.** *A set* $S \subset \mathbf{R}$ *is called a* **neighbourhood (nbd.)** *of a point* $p \in \mathbf{R}$ *if there exists an open interval* $]p - \varepsilon, p + \varepsilon[$ *for some* $\varepsilon > 0$, *such that*

$$p \in\ ]p - \varepsilon,\ p + \varepsilon\,[\ \subset S.$$

*Equivalently, a set S is a* **neighbourhood** *of a point p if there exists an open interval I such that* $p \in I \subset S$.

**Illustrations**

1. *The set* $\mathbf{R}$ *of real numbers is a nbd. of all its points.*

   For any real number $x$, we have

   $$x \in\ ]x - \varepsilon,\ x + [\subset \mathbf{R},\ \ \varepsilon > 0.$$

   (Remember $\mathbf{R}$ as the real line).

2. *The open interval* $]a, b[$ *is a nbd. of all its points*, since

   $$x \in\ ]a, b[\ \subset\ ]a, b[\quad \forall\quad x \in\ ]a, b[.$$

3. *The half open interval* $]a, b]$ *is a nbd. of all its points except* ***b***, since for any $\varepsilon > 0$

   $$b \in\ ]b - \varepsilon, b + \varepsilon\,[\ \not\subset\ ]\,a, b]. \qquad (\because\ b + \varepsilon > b)$$

   Similarly, $[a, b[$ is a *nbd.* of all its points *except a* and $[a, b]$ is a *nbd.* of all its points *except a* and *b*.

4. ***The set Z of integers is not a nbd. of any of its points,*** since for any integer $p$ and any $\varepsilon > 0$,

   $$p \in\ ]p - \varepsilon, p + \varepsilon[\ \not\subset \mathbf{Z}.$$

5. ***Each of the following open intervals*** :

   $$]-1, 1[,\ ]-\tfrac{1}{2}, \tfrac{1}{2}[,\ ]-\tfrac{1}{3}, \tfrac{1}{3}[,\ \ldots,\ ]-\tfrac{1}{n}, \tfrac{1}{n}[,\ \ldots$$ *is a nbd.* of 0.

6. ***For any*** $p \in \mathbf{R}$, $N = \{\,p\,\}$ **is not** ***a nbd. of p***, since for any $\varepsilon > 0$, $]\,p - \varepsilon, p + \varepsilon\,[\ \not\subset N$.

7. ***The set*** **Q** ***of rational numbers is*** **not** ***a nbd. of any of its points***, for if $]a, b[$ is any open interval around a point $p \in \mathbf{Q}$, then

   $$p \in\ ]a, b[\ \not\subset \mathbf{Q},$$

   since between any two real numbers $a$, $b$; there is always an irrational number which does not belong to **Q**.

$$p \in \,]p-\varepsilon, p+\varepsilon[\, \subset M_i \text{ for } i = 1, 2, \ldots, n$$

$$\Rightarrow \quad p \in \,]p-\varepsilon, p+\varepsilon[\, \subset M_1 \cap M_2 \cap \ldots \cap M_n.$$

Hence $M_1 \cap M_2 \cap \ldots \cap M_n$ is a *nbd.* of $p$.

**Example 4.** *Define neighbourhood of a point of* **R**. *Show that the intersection of two neighbourhoods of a point of* **R** *is again a neighbourhood of the same point. What happens to the conclusion of the statement if arbitrary collection of neighbourhoods is taken ?*

[D.U., Maths (H) 1996]

**Solution.** (*i*) See Example 2.

(*ii*) Let $G_n = \,]-1/n, 1/n[\; \forall\, n \in N$. Then $G_n$ is a *nbd.* of 0 for each $n$. Clearly $\bigcap\limits_{n=1}^{\infty} G_n = \{0\}$, where $\{0\}$ is not a *nbd.* of 0. Thus it follows that the intersection of an arbitrary collection of *nbds.* of a point may not be a *nbd.* of that point.

**Example 5.** *Define neighbourhood of a point and justify that arbitrary intersection of the neighbourhoods of a point may not be a neighbourhood of the point.*

[B.C.A. Sample Problem]

**Hint.** See Example 4.

**Example 6.** *Define a neighbourhood of a point. Show that the intersection of the family of all neighbourhoods of a point* $x$ *is* $\{x\}$.

[D.U., Maths (H) 1999 ; B.Sc. (G) 1995]

**Solution.** A subset $S \subset \mathbf{R}$ is called a *neighbourhood* of a point $p \in \mathbf{R}$ if there exists some $\varepsilon > 0$ such that $]p-\varepsilon, p+\varepsilon[\, \subset S$.

Let $\{N_x\}$ be a family of all neighbourhoods of a point $x$. We have to show that $\cap N_x = \{x\}$.

Consider any point $y \neq x$. Let $\varepsilon = |x-y|$. Then $]x-\varepsilon, x+\varepsilon[$ is a *nbd.* of $x$ and $y \notin \,]x-\varepsilon, x+\varepsilon[$, since

$$y \in \,]x-\varepsilon, x+\varepsilon[\, \Rightarrow x-\varepsilon < y < x+\varepsilon \Rightarrow |x-y| < \varepsilon, \text{ a contradiction.}$$

Since $y \notin \,]x-\varepsilon, x+\varepsilon[$, $y \notin \cap N_x$. Hence $\cap N_x = \{x\}$.

**Example 7.** (*i*) *Can a set* $N$ *be a neighbourhood of any point of* $\mathbf{R} - N$ ? (*ii*) *Can a non-empty finite set be a neighbourhood of any of its points ?*

[D.U., Maths (H) 1999, 97]

**Solution.** (*i*) We shall show that $N$ cannot be a neighbourhood of any point of $\mathbf{R} - N$. Let $x \in \mathbf{R} - N$. Then $x \notin N$. Suppose $N$ is a *nbd.* of $x$. Then there exists some $\varepsilon > 0$ such that $x \in \,]x-\varepsilon, x+\varepsilon[\, \subset N \Rightarrow x \in N$, which is a contradiction. Hence the assertion made above.

(*ii*) The answer of the second part is also No. Let $x \in N$, where $N$ is a non-empty finite set. If $N$ is *nbd.* of $x$, then

$$x \in \,]x-\varepsilon, x+\varepsilon[\, \subset N \text{ for some } \varepsilon > 0.$$

This is impossible, since N is a finite set and

**Example 8.** *Give an example of each of the following :*

*(a) a set which is a nbd. of each of its points,*

*(b) a set which is not a nbd. of any of its points,*

*(c) a set which is a nbd. of each of its points with the exception of one point,*

*(d) a set which is not an interval but is a nbd. of each of its points.*

**Solution.** (*a*) The set **R** of real numbers or any open interval $]a, b[$ is a *nbd.* of each of its points.

(*b*) The set **Z** of integers is not a *nbd.* of any of its points.

(*c*) $[a, b[$ is a *nbd.* of all its points except the point *a*.

(*d*) $G = ]1, 2[ \cup ]3, 4[$ is a *nbd.* of each of its points and *G* is not an interval, since *G* actually is $]1, 4[$ out of which $[2, 3]$ has been removed.

## 2.2 Open Sets

**Definition.** *A set $S \subset \mathbf{R}$ is called an* **open set** *if it is a nbd. of each of its points.*

*In other words : A set $S \subset \mathbf{R}$ is called an* **open set** *if for each $p \in S$, there exists some $\varepsilon > 0$ such that*

$$p \in ]p - \varepsilon, p + \varepsilon[ \subset S \quad \forall \quad p \in S.$$

**Note.** It may be noted that a set *S* is *not* an open set if *S* is not a *nbd.* of *even one* of its points.

**Illustrations**

1. *The set* **R** *of real numbers is an open set.*
2. *Every open interval $]a, b[$ is an open set.*
   Thus $]-\infty, a[$ and $]b, +\infty[$ are also open sets.
3. $[a, b[$ is *not* an open set, since $[a, b[$ is a *nbd.* of all its points *except a*.
   Similarly, $]a, b]$, $[a, b]$ are *not* open sets.
4. *The set* **Z** *of integers is not an open set,*
   since **Z** is not a *nbd.* of each of its points.
5. $G = ]1, 2[ \cup ]3, 4[$ is an open set, as *G* is a *nbd.* of each of its points.
6. $\left] -\dfrac{1}{n}, \dfrac{1}{n} \right[$ is an open set for each $n \in \mathbf{N}$.
7. *The empty set $\phi$ is an open set.*
8. The singleton set $\{x\}$ is **not** an open set, since for any $\varepsilon > 0$, $]x - \varepsilon, x + \varepsilon[ \not\subset \{x\}$
9. *The set* **Q** *of rational numbers is* **not** *an open set, since* **Q** *is not a nbd. of each of its points.*
10. ***The set* R ~ Q *of all irrational numbers is not an open set.***
11. **The set** $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ **is not an open set, as** *S* **is not a** *nbd.* **of each of its points.** For example, $]1 - \varepsilon, 1 + \varepsilon[$ **contains an infinite number of irrational numbers and so** $]1 - \varepsilon, 1 + \varepsilon[ \not\subset S$ **for each $\varepsilon > 0$.**

**Theorem 1.** *Prove that the union of an arbitrary family of open sets is an open set.* [D.U., Maths (H) 1997, 92 ; B.Sc. (G) 96]

**Proof.** Let $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of open sets.

Let $G = \bigcup_{\lambda \in \Lambda} G_\lambda$. We have to show that *G* is an open set.

Let *x* be *any* element of $G = \cup G_\lambda$.

Then $x \in G_\lambda$ for some $\lambda \in \Lambda$.

Since $G_\lambda$ is given to be an open set and $x \in G_\lambda$, so $G_\lambda$ is a *nbd.* of *x*.

Thus there exist some $\varepsilon > 0$ such that

$$x \in ]x - \varepsilon, x + \varepsilon[ \subset G_\lambda$$

$$\Rightarrow \quad x \in ]x - \varepsilon, x + \varepsilon[ \subset G_\lambda \subset \bigcup_{\lambda \in \Lambda} G_\lambda = G$$

$$\Rightarrow \quad x \in ]x - \varepsilon, x + \varepsilon[ \subset G \quad \forall \; x \in G$$

$$\Rightarrow \quad G \text{ is a } nbd. \text{ of } x, \;\; \forall \; x \in G.$$

It follows that *G* is a *nbd.* of each of its points and consequently $G = \cup G_\lambda$ is an open set.

**Theorem 2.** *Prove that the intersection of a finite number of open sets is an open set.* [D.U., Maths (H) 1992 ; B.Sc. (G) 96]

**Proof.** *First of all we show that the intersection of two open sets $G_1$ and $G_2$ is an open set.*

Let *x* be *any* point of $G_1 \cap G_2$. Then $x \in G_1 \cap G_2$

$\Rightarrow x \in G_1$ and $x \in G_2$

$\Rightarrow G_1$ and $G_2$ are *nbds.* of *x* ($\because$ $G_1$ and $G_2$ are open sets)

$\Rightarrow G_1 \cap G_2$ is a *nbd.* of *x* (by Example 2)

$\Rightarrow G_1 \cap G_2$ is a *nbd.* of *x* for each $x \in G_1 \in G_2$.

Thus $G_1 \cap G_2$ is an open set.

Now we consider three open sets $G_1, G_2, G_3$. Then

$$G_1 \cap G_2 \cap G_3 = (G_1 \cap G_2) \cap G_3 .$$

R.H.S. being the intersection of two open sets *viz.* $G_1 \cap G_2$ and $G_3$ is an open set and so $G_1 \cap G_2 \cap G_3$ is an open set.

Proceeding in a similar way, if $G_1, G_2, \ldots\ldots, G_n$ are a finite number of open sets, then

$$G_1 \cap G_2 \cap \ldots \cap G_n \text{ is an open set.}$$

**Remark.** *The intersection of an arbitrary family of open sets* ***may not*** *be an open set.*

Let us consider $G_n = ]-1/n, 1/n[ \; \forall \; n \in \mathbf{N}$. Then

$$G_1 = ]-1, 1[, G_2 = ]-1/2, 1/2[, G_3 = ]-1/3, 1/3[, \ldots$$

Since every open interval is an open set, therefore $\{G_n : n \in \mathbf{N}\}$ is a family of infinite number of open sets.

**Clearly.** $G_1 \cap G_2 = ]-1, 1[ \cap ]-1/2, 1/2[ = ]-1/2, 1/2[$,

$G_1 \cap G_2 \cap G_3 = ]-1/2, 1/2[ \cap ]-1/3, 1/3[ = ]-1/3, 1/3[$

and so on. Thus at every step, the size of the open interval is reducing and getting closer to 0.

**Indeed** $\bigcap_{n=1}^{\infty} G_n = \{0\}$, which is not an open set,

since for any $\varepsilon > 0$, $]0-\varepsilon, 0+\varepsilon[ = ]-\varepsilon, \varepsilon[ \not\subset \{0\}$.

**Ex. 1.** *Prove that the union of an arbitrary family of* ***open sets*** *is an open set and the intersection of a finite number of* ***open sets is an open set.*** *Give an example to show that the intersection of an arbitrary family of open sets may not be an open set.* [D.U., B.Sc. (G) 1996]

**Hint.** Prove Theorems 1 and 2 and Remark given above.

**Ex. 2.** *Prove that the intersection of two open sets is open. Does the above result hold for the intersection of an arbitrary family of open sets ? Illustrate your answer by an example.* [D.U., Maths (H) 1993]

**Hint.** See Example 12 ahead.

**Ex. 3.** Let $I_n = \left]x - \dfrac{1}{n}, x + \dfrac{1}{n}\right[$, $n = 1, 2, \ldots$ *Prove that* $I_n$ *is an open set for each positive integer* $n$ *and* $\bigcap_{n=1}^{\infty} I_n = \{x\}$. [D.U., Maths (H) 1998]

**Hint.** Since each open interval is an open set, $]x - 1/n, x + 1/n[$ is an open set for $n = 1, 2, 3, \ldots$. As shown above, $\bigcap_{n=1}^{\infty} \left]-\dfrac{1}{n}, \dfrac{1}{n}\right[ = \{0\}$. Similarly,

$$\bigcap_{n=1}^{\infty} \left]x - \frac{1}{n}, x + \frac{1}{n}\right[ = \{x+0\} = \{x\}.$$

**Ex. 4.** *Is every infinite set open ? Justify your answer.* [D.U., Maths (H) 1999, 97]

**Hint.** Every infinite set need not be an open set. For example,

(*i*) the set **Z** of integers is an infinite set, which is not an open set.

(*ii*) $S = \left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\right\}$ is an infinite set, which is not an open set.

**5.** *Can a non-empty finite set be an open set ?*

No [See Example 7(*ii*)].

## 2.3 Closed Sets

**Definition.** *A set* $S \subset \mathbf{R}$ *is called a* **closed set** *if and only if its complement* $\mathbf{R} \sim S$ *is an* **open set**.

*S is closed* $\Leftrightarrow \mathbf{R} \sim S$ *is open.*

**Illustrations**

1. *The set* **R** *of real numbers is a closed set,*
as $\mathbf{R} \sim \mathbf{R} = \phi$ is an open set.
2. *The empty set* $\phi$ *is a closed set,*
as $\mathbf{R} \sim \phi = \mathbf{R}$ is an open set.
3. *The set* **R** *of real numbers is both open and closed.*
*The empty set* $\phi$ *is both open and closed.*
4. *Every closed interval* $[a, b]$ *is a closed set,*
$\because \quad \mathbf{R} \sim [a, b] = ]-\infty, a[ \cup ]b, +\infty[$
and R.H.S. being the union of two open sets is an open set.
5. $[a, b[$ is not a closed set,
$\because \quad \mathbf{R} \sim [a, b[ = ]-\infty, a[ \cup [b, +\infty[$
is not an open set as the R.H.S. is a *nbd.* of each of its points **except *b*.**
Similarly, $]a, b]$, $]a, b[$ are not closed sets.
6. $[0,3] \cap [1,2]$ is a closed set,
since $[0,3] \cap [1,2] = [1,2]$, which is a closed set.
7. $F = [1, 2] \cup [2, 4]$ is a closed set, since $F = [1, 4]$ is a closed set.
8. **Q** *is* **not** *a closed set*, since $\mathbf{R} \sim \mathbf{Q}$ being the set of all irrational numbers is not an open set. *Thus the set* **Q** *of rational numbers is neither a closed set nor an open set.*

**Theorem 3.** (*a*) *The union of a finite number of closed sets is a closed set.* [D.U., B.Sc. (G) 1994 ; Maths (H) 1992]

(*b*) *The intersection of an arbitrary family of closed sets is a closed set.* [D.U., B.Sc. (G) 1995, 94 ; Maths (H) 1995, 93]

(*c*) *Give an example to show that an arbitrary union of* ***closed sets*** *may not be a closed set.* [D.U., B.Sc. (G) 1995]

**Proof.** (*a*) Let $F_1, F_2, \ldots, F_n$ be a finite number of closed sets so that $\mathbf{R} \sim F_1, \mathbf{R} \sim F_2, \ldots, \mathbf{R} \sim F_n$ are open sets.

$\therefore \quad (\mathbf{R} \sim F_1) \cap (\mathbf{R} \sim F_2) \cap \ldots \cap (\mathbf{R} \sim F_n)$ is an open set. (Theorem 2)

By De Morgan's law,

$\mathbf{R} \sim (F_1 \cup F_2 \cup \ldots \cup F_n) = (\mathbf{R} \sim F_1) \cap (\mathbf{R} \sim F_2) \cap \ldots \cap (\mathbf{R} \sim F_n)$

$\Rightarrow \quad \mathbf{R} \sim (F_1 \cup F_2 \cup \ldots \cup F_n)$ is an open set.

Hence $F_1 \cup F_2 \cup \ldots \cup F_n$ is a closed set.

(*b*) Let $\{F_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of closed sets.

**Example 11.** *Comment on the following statements :*

*(a) Can a finite non-empty set be open ?*

**Ans.** Yes. An empty set is a finite set, which is also an open set.

*(b) Can a finite non-empty set be open ?*

**Ans.** No, for if $S$ is a non-empty finite open set, then for any $x \in S$ and any $\varepsilon > 0$] $x - \varepsilon, x + \varepsilon$[ $\not\subset S$, since the interval ]$x - \varepsilon, x + \varepsilon$[ necessarily contains an infinite number of points and $S$ is a finite set.

*(c) Is every infinite set open ?*

**Ans.** No. The set **Z** of integers is an infinite set, which is not an open set.

*(d) Is the intersection of any arbitrary family of open sets an open set ?*

**Ans.** No. $G_n = ]-1/n, 1/n[\ \forall\ n \in \mathbf{N}$ is a family of infinite number of open sets and $\bigcap_{n=1}^{\infty} G_n = \{0\}$, which is not an open set.

**Example 12.** *Prove that intersection of two open sets is open. Is the result true for arbitrary intersection of open sets ? Justify your answer.*

[D.U., B.A. (P) 1996, 95]

**Solution.** Let $G$ and $H$ be two open sets. We shall prove that $G \cap H$ is an open set *i.e.*, $G \cap H$ is a *nbd.* of each of its points. Let $x$ be any point of $G \cap H$. Now $x \in G \cap H \Rightarrow x \in G$ and $x \in H$

$\Rightarrow G$ and $H$ are *nbds.* of $x$ ($\because$ $G$ and $H$ are open sets)

$\Rightarrow G \cap H$ is a *nbd.* of $x$ [See Example 2]

Thus $G \cap H$ is a *nbd.* of all its points and so $G \cap H$ is an open set.

*The intersection of an arbitrary family of open sets may not be an open set.*

Let $G_n = ]-1/n, 1/n[\ \forall\ n \in N$. Then $\{G_n : n \in N\}$ is a family of infinite number of open sets. It follows that $\bigcap_{n=1}^{\infty} G_n = \{0\}$, which is not an open set.

**Example 13.** *Prove that the union of two closed sets is closed. Does the above result hold for the union of an arbitrary family of closed sets ? Illustrate your answer by an example.*

[D.U., Maths (H) 1993]

**Solution.** Let $A$ and $B$ be two closed sets. We shall prove that $A \cup B$ is a closed set i.e., its complement $\mathbf{R} - (A \cup B)$ is an open set.

Now $\qquad \mathbf{R} - (A \cup B) = (\mathbf{R} - A) \cap (\mathbf{R} - B)$. $\qquad$ ...(*i*)

Since $A$ and $B$ are closed sets, $\mathbf{R} - A$ and $\mathbf{R} - B$ are open sets. Since the intersection of two open sets is an open set, so, $(\mathbf{R} - A) \cap (\mathbf{R} - B)$ is an open set. Using (*i*), $\mathbf{R} - (A \cup B)$ is an open set. Hence $A \cup B$ is a closed set. The union of an arbitrary family of closed sets may not be a closed set.

Let $F_n = [\, 1/n,\ 2] \ \forall\ n \in \mathbf{N}$. Since every closed interval is a closed set, $\{F_n : n \in \mathbf{N}\}$ is a family of infinite number of closed sets. We have

$$\bigcup_{n=1}^{\infty} F_n = ]\,0,\ 2],$$ which is not a closed set.

**Example 14.** *Show that every open set is a union of open intervals.*

**Solution.** Let $G$ be any open set. By definition, for each $x \in G$, there exists some $\varepsilon_x > 0$, such that $x \in\ ]\,x-\varepsilon_x,\ x+\varepsilon_x\,[\ \subset G$. It follows that

$$G \subset \bigcup_{x \in G} ]\,x-\varepsilon_x,\ x+\varepsilon_x\,[\ \subset G.$$

Hence $\quad G = \bigcup_{x \in G} ]\,x-\varepsilon_x,\ x+\varepsilon_x\,[$, which proves the result.

## 2.4 Limit Points

**Definition 1.** *A real number $p$ is said to be a* **limit point** (*or* **accumulation point**) *of a set $S \subset \mathbf{R}$ if every nbd. of $p$ contains a point of $S$ different from $p$.*

*In other words, a real number $p$ is said to be a* **limit point** *of a set $S \subset \mathbf{R}$ if for each $\varepsilon > 0$, there exists at least one point $p_1 \in S$ such that*

$$p_1 \in\ ]\,p-\varepsilon,\ p+\varepsilon\,[ \ \textit{and}\ p_1 \neq p.$$

**Important Note.** It may be observed that *a real number $p$ is* **not a limit point** *of a set $S$ if there exists* **some** $\varepsilon > 0$ *such that* $]p-\varepsilon, p+\varepsilon[$ *contains no point of $S$ different from $p$ i.e.,*

*either* $]\,p-\varepsilon, p+\varepsilon\,[\ \cap S = \{p\}$ *or* $]\,p-\varepsilon, p+\varepsilon\,[\ \cap S = \phi$.

**Definition 2.** *The set of all limit points of a set $S \subset \mathbf{R}$ is called the* **derived set** *of $S$ and is denoted as $S'$.*

$x \in S' \Leftrightarrow x$ *is a limit point of $S$.*

**Illustrations**

1. *Every point of* $\mathbf{R}$ *is a limit point of* $\mathbf{R}$, since for each $p \in \mathbf{R}$ and for each $\varepsilon > 0$, $]\,p-\varepsilon, p+\varepsilon\,[$ must contain a point $p_1 \in \mathbf{R}$ such that $p_1 \neq p$. Indeed $\mathbf{R}' = \mathbf{R}$.
2. *Every real number is a limit point of* $\mathbf{Q}$ (the set of rational numbers), since for each $p \in \mathbf{R}$ and for each $\varepsilon > 0$, $]\,p-\varepsilon, p+\varepsilon\,[$ contains a rational number $q \in \mathbf{Q}$ such that $q \neq p$. Indeed $\mathbf{Q}' = \mathbf{R}$.
3. 0 *is the only limit point of the set*

$$S = \left\{ 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots, \tfrac{1}{n}, \ldots \right\}.$$

Notice that $]\,0-\frac{1}{10}, 0+\frac{1}{10}\,[$ contains $\frac{1}{11}, \frac{1}{12}, \ldots$ etc., which are all different from 0. In fact, if $\varepsilon > 0$ is any arbitrary number, $]0-\varepsilon, 0+\varepsilon\,[\ =\ ]-\varepsilon, \varepsilon\,[$ must contain some $1/n$ $(n > 0)$. Thus 0 is a limit [illegible] of $S$. Now we show if $p \neq 0$, then $p$ cannot be a limit point of $S$.

(*i*) If $p < 0$, then $]-\infty, 0\,[$ is a *nbd.* of $p$ which does not contain any point of $S$.

(*ii*) If $p > 1$, then $]1, \infty\,[$ is a *nbd.* of $p$ which does not contain any point of $S$.

(*iii*) If $p = 1$ then $]\,1-\frac{1}{2}, 1+\frac{1}{2}\,[\ =\ ]\,\frac{1}{2}, \frac{3}{2}\,[$ is a *nbd.* of 1 which does not contain any point of $S$ except 1 itself.

(*iv*) Let $0 < p < 1$ and $p \in S$, say $p = \frac{1}{2}$. Then

$$]\,\tfrac{1}{2}-\tfrac{1}{10}, \tfrac{1}{2}+\tfrac{1}{10}\,[\ =\ ]\,\tfrac{2}{5}, \tfrac{3}{5}\,[$$

is a *nbd.* of $\frac{1}{2}$ which does not contain any point of $S$ other than $\frac{1}{2}$.

(*v*) Let $0 < p < 1$ and $p \notin S$. Then $\exists$ a positive integer $m$ such that $m < 1/p < m+1 \Rightarrow (m+1)^{-1} < p < m^{-1}$.

Thus $]\,(m+1)^{-1}, m^{-1}\,[$ is a *nbd.* of $p$ which does not contain any point of $S$.

Hence 0 is the only limit point of $S$.

4. 1 *is the only limit point of the set*

$$S = \left\{ 1, 1\tfrac{1}{2}, 1\tfrac{1}{3}, 1\tfrac{1}{4}, 1\tfrac{1}{5}, \ldots \right\}.$$

Clearly, $S' = \{1\}$. Notice that $1 \in S$.

5. 1 *and* $-1$ *are the only limit points of the set*

$$S = \left\{ 1, -1, 1\tfrac{1}{2}, -1\tfrac{1}{2}, 1\tfrac{1}{3}, -1\tfrac{1}{3}, \ldots \right\}.$$

Notice that $1, -1$ both belong to $S$ and $S' = \{1, -1\}$.

6. *The set* $\mathbf{Z}$ *of integers has no limit point.*

If $m$ be any integer, then $]\,m-\frac{1}{2}, m+\frac{1}{2}\,[$ is a *nbd.* of $m$ which does not contain any point of $\mathbf{Z}$ other than $m$.

If $p \in \mathbf{R} \sim \mathbf{Z}$, then we can find some integer $i$ such that

$$i < p < i+1.$$

Consequently $]\,i, i+1\,[$ is a *nbd.* of $p$ which does not contain any point of $\mathbf{Z}$. Hence $\mathbf{Z}$ has no limit point *i.e.*, $\mathbf{Z}' = \phi$.

7. $\{]\,0, 1\,[\,\}' = [0, 1]$.

In other words, *every element of* $[0, 1]$ *is a limit point of* $]0, 1[$.

Let $\varepsilon > 0$ be arbitrary. Let $0 < p < 1$. Then

$$]-\varepsilon, \varepsilon\,[\,,\ ]\,p-\varepsilon, p+\varepsilon\,[\,,\ ]\,1-\varepsilon, 1+\varepsilon\,[$$

are *any nbds.* of 0, $p$ and 1, respectively, each of which contains a point of $]0, 1[$ other than 0, $p$ and 1, respectively. Thus every point of $[0, 1]$ is a limit point of $]0, 1[$.

Similarly, we obtain

$$\{\,[0, 1[\,\}' = [0,1],\ \{\,]0,1]\,\}' = [0, 1],\ \{\,[0, 1]\,\}' = [0,1].$$

**Remarks**

1. *A set may or may not have a limit point.*

For example, **R** has limit points but **Z** has no limit point.

2. *Limit point of a set may or may not belong to the set.*

For example, every limit point of **R** belongs to **R** and the only limit point 0 of the set $S = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$ does not belong to the set *i.e.*, $0 \notin S'$.

3. We have seen above that 0 is the only limit point of the set $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$. Here the points of the set $S$ are closely scattered around the limit point 0. *Thus it appears that such points,* ***around which are scattered closely,*** *the elements of a given set S,* ***have a strong possibility of becoming the limit points of the set S.***

**Ex.** ***Define limit point of a set and derived set of a set.***

## EXAMPLES

**Example 15.** *Find the limit points of each of the following sets :*

(*a*) $A = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \right\}$. (*b*) $B = \left\{ 1 + \frac{1}{n} : n \in \mathbf{N} \right\}$.

(*c*) $C = \left\{ 1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3} ... \right\}$.

(*d*) $D = \{x : 0 \le x < 1\}$.

(*e*) $E = \left\{ \frac{1}{m} + \frac{1}{n} : m \in \mathbf{N}, n \in \mathbf{N} \right\}$.

(*f*) $F = \{3^{-n} + 5^{-n} : n \in \mathbf{N}\}$.

**Solution.** (*a*) The only limit point of $A$ is 0 *i.e.*, $A' = \{0\}$. Notice that $0 \notin A$.

(*b*) We have $B = \left\{ 2, 1\frac{1}{2}, 1\frac{1}{3}, 1\frac{1}{4}, ... \right\}$.

The only limit point of $B$ is 1 *i.e.*, $B' = \{1\}$. Notice that $1 \notin B$.

(*c*) The set $C = \left\{ 1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, ... \right\}$ has only two limit points 1, − 1. Notice that $1 \in C$ and $-1 \in C$.

We have $C' = \{1, -1\}$.

(*d*) We observe that $D = [0, 1[$ and so $D' = [0, 1]$.

(*e*) First we write all the elements of the set

$$E = \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbf{N} \right\}.$$

For $m = 1$, the elements of $E$ are :

$$1 + 1, 1\frac{1}{2}, 1\frac{1}{3}, 1\frac{1}{4}, ... \quad ...(1)$$

For $m = 2$, the elements of $E$ are :

$$\frac{1}{2} + 1, \frac{1}{2} + \frac{1}{2}, \frac{1}{2} + \frac{1}{3}, \frac{1}{2} + \frac{1}{4}, ... \quad ...(2)$$

For $m = 3$, the elements of $E$ are

$$\frac{1}{3} + 1, \frac{1}{3} + \frac{1}{2}, \frac{1}{3} + \frac{1}{3}, \frac{1}{3} + \frac{1}{4}, ... \quad ...(3)$$

and so on.

It is clear that

1 is the limit point of the elements in row (1),

$\frac{1}{2}$ is the limit point of the elements in row (2),

$\frac{1}{3}$ is the limit point of the elements in row (3),

and so on.

Thus $1, \frac{1}{2}, \frac{1}{3}, ...., \frac{1}{n}, ....$ are all limit points of $E$.

Since $E$ contains an infinite number of fractions lying between 0 and 1, so each *nbd*. of 0 surely contains a positive fraction and as such 0 is also a limit point of $E$.

Hence $E' = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\} \cup \{0\}$.

(*f*) We have $F = \left\{ \frac{1}{3^n} + \frac{1}{5^n} : n \in \mathbf{N} \right\}$.

As discussed in part (*e*), we observe that

$$F' = \left\{ \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, ... \right\} \cup \left\{ \frac{1}{5}, \frac{1}{5^2}, \frac{1}{5^3}, ... \right\} \cup \{0\}.$$

**Example 16.** *Give an example of a set whose* ***derived set is***

(*i*) *void*

**Ans.** If **Z** is the set of integers, then $\mathbf{Z}' = \phi$.

(*ii*) *subset of the given set*

**Ans.** Let $S = \left\{ 1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, .... \right\}$.

Then $S' = \{1, -1\} \subset S$.

(*iii*) *superset of the given set*

**Ans.** Let $S = ]0, 1[$. Then $S' = [0, 1]$ and $S' \supset S$.

(*iv*) ***neither a subset nor a superset of the given set***

**Ans.** Let $S = \left\{ 1, \frac{1}{2}, \frac{1}{3}, ... \frac{1}{n}, ... \right\}$. Then $S' = \{0\}$.

$S'$ is neither a subset nor a superset of $S$.

(*v*) *same as the given set*

**Ans.** $\mathbf{R}' = \mathbf{R}$ or $[a, b]' = [a, b]$.

**Example 17.** *Give an example of a set which has*

*(i) no limit point*

**Ans.** The set **Z** of integers has no limit point.

*(ii) unique limit point*

**Ans.** $S = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \right\}$ has 0 as the unique limit point.

*(iii) two limit points*

**Ans.** $S = \left\{ 1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots \right\}$

has two limit points *viz.* 1 and − 1.

*(iv) three limit points*

**Ans.** $S = \left\{ 1, 2, 3, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 1\frac{1}{3}, 2\frac{1}{3}, 3\frac{1}{3}, \ldots \right\}$ has three limit points *viz.* 1, 2, 3

*(v) infinite number of limit points*

**Ans.** The set **R** of real numbers has infinite number of limit points. In fact every real number is a limit point of **R**.

*(vi) a set whose every point is a limit point of the set*

**Ans.** The set **R** of real numbers.

*(vii) a set none of whose points is a limit point of the set*

**Ans.** The set **Z** of integers.

*(viii)* ***A*** *set with only* $\sqrt{3}$ *as its limit point.*

**Ans.** $S = \left\{ \sqrt{3} + \frac{1}{n} : n \in \mathbf{N} \right\}$.

**Example 18.** *Define limit point of a set. Show that the set* **N** *of natural numbers has no limit point.* [D.U., B.Sc. (G) 1988]

**Solution.** Let $m$ be any natural number.

Then $]\, m - \frac{1}{2}, m + \frac{1}{2}\, [$ is a *nbd.* of $m$ which does not contain any point of **N** different from $m$.

Let $m \in \mathbf{R} \sim \mathbf{N}$ *i.e.*, $m$ is any real number but not a natural number. Then we can find some integer $j$ such that $j < m < j + 1$. Consequently $]\, j, j + 1\, [$ is a *nbd.* of $m$, which does not contain any point of **N**. Hence **N** has no limit point *i.e.*, $\mathbf{N}' = \phi$.

**Example 19.** *Find the limit points of the set*

$$S = \left\{ \frac{1}{2m} + \frac{1}{3n} : m,\ n = 1, 2, 3, \ldots \right\} \quad \text{[D.U., Maths (H) 1996]}$$

**Solution.** Refer to Example 15(*f*). The set of all limits points of the given set is

$$S' = \left\{ \frac{1}{2m} : m \in \mathbf{N} \right\} \cup \left\{ \frac{1}{3n} : n \in \mathbf{N} \right\} \cup \{0\}.$$

**Example 20.** *Show that (i) every rational number is a limit point of* **R** ~ **Q** *(set of all irrational numbers), (ii) every irrational number is a limit point of* **Q**.

**Solution.** (*i*) Let $x \in \mathbf{Q}$ be arbitrary. Let $\varepsilon > 0$ be given. Then there exists an irrational number $s$ ($s \in \mathbf{R} \sim \mathbf{Q} \Rightarrow s \neq x$) such that

$$x - \varepsilon < s < x + \varepsilon \text{ i.e., } s \in \,]\, x - \varepsilon, x + \varepsilon\, [,\ s \neq x.$$

Hence $x$ is a limit point of **R** ~ **Q**.

(*ii*) The proof is on the same lines.

**Example 21.** *Give an example each of :*

*(i) a bounded set of real numbers*

*(ii) an unbounded set of real numbers*

*Also find the set of limit points of both the sets.*

**Solution.** (*i*) $S = \left\{ 1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots \right\}$

is a bounded set of real numbers. Its limit points are 1 and − 1.

(*ii*) $S = \left\{ 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots \right\}$

is an unbounded set of real numbers. 0 is the only limit point of $S$.

**Example 22.** *Find the derived sets of the following :*

(*i*) $A = \left\{ 1 + \frac{(-1)^n}{n} : n \in \mathbf{N} \right\}$, (*ii*) $B = \left\{ (-1)^n + \frac{1}{n} : n \in \mathbf{N} \right\}$,

(*iii*) $C = \left\{ \frac{1 + (-1)^n}{n} : n \in \mathbf{N} \right\}$, (*iv*) $D = \left\{ 3^n + \frac{1}{3^n} : n \in \mathbf{N} \right\}$.

**Solution.** $A' = \{1\}, B' = \{-1, 1\}, C' = \{0\}, D' = \phi$.

**Example 23.** *Let A be a non-empty subset of* **R** *and* $K = \sup A$. *If* $K \notin A$, *prove that K is a limit point of A.*

**Solution.** Let $\varepsilon > 0$ be arbitrary. Since $K = \text{l.u.b.} A$, $K - \varepsilon$ is not an upper bound of $A$ and so there exists some $a \in A$ such that $K - \varepsilon < a$, $a \neq K$. Also $a \in A \Rightarrow a \leq K < K + \varepsilon$. Thus $K - \varepsilon < a < K + \varepsilon$ and so for each $\varepsilon > 0$, $]\, K - \varepsilon, K + \varepsilon\, [$ contains a point $a \in A$ such that $a \neq K$. Hence $K$ is a limit point of $A$.

**Ex.** *Let A be a non-empty subset of* **R** *and* $k = \inf A$. *If* $k \notin A$, ***prove*** *that k is a limit point of A.*

**Hint.** $k + \varepsilon > a$ for some $a \in A$. Also $k - \varepsilon < k \leq a$.

**Example 24.** *If x is a limit point of a set* $S \subset \mathbf{R}$, *is x a limit point of* $S \sim \{x\}$ *also ? Justify your answer.* [D.U., Maths (H) 1999]

**Solution.** We shall prove that $x$ is a limit point of $S - \{x\}$. Since $x$ is a limit point of $S$, each *nbd.* $N$ of $x$ contains some point $y \in S$ such that

$$y \neq x \quad \Rightarrow \quad y \in S - \{x\}.$$

Hence each *nbd.* $N$ of $x$ contains some point $y \in S-\{x\}$ such that $y \neq x$. Hence $x$ is a limit point of $S-\{x\}$.

**Example 25.** *Let* $A \subset \mathbf{R}$ *and* $G$ *an open subset of* $\mathbf{R}$. *Show that* $G \cap A = \phi \Rightarrow G \cap A' = \phi$.

**Solution.** Let, if possible, $p \in G \cap A'$. Then $p \in G$ and $p \in A'$. Since $G$ is an open set and $p \in A'$, $p$ is a limit point of $A$ and so every *nbd.* of $p$ (in particular, $G$) contains a point of $A$, say $a \in A$ and $a \neq p$ *i.e.*, $a \in G \cap A \Rightarrow G \cap A \neq \phi$, a contradiction. Hence $G \cap A' = \phi$.

**Example 26.** *If* $S$ *and* $T$ *are sets of real numbers, then show that*

(*a*) $S \subset T \Rightarrow S' \subset T'$, (*b*) $(S \cup T)' = S' \cup T'$,

(*c*) $(S \cap T)' \subset S' \cap T'$. [D.U., B.Sc. (G) 1997]

*Give an example to show that* $(S \cap T)'$ *and* $S' \cap T'$ ***may not be equal.***

**Solution.** (*a*) Let $S \subset T$. We shall prove that $S' \subset T'$.

Let $x \in S'$ so that $x$ is a limit point of $S$. **Thus for each** $\varepsilon > 0$, $]x-\varepsilon, x+\varepsilon[$ contains a point $x_1 \in S$ such that $x_1 \neq x$. **Since** $S \subset T$, $x_1 \in S \Rightarrow x_1 \in T$. Consequently, $]x-\varepsilon, x+\varepsilon[$ **contains a point** $x_1 \in T$, $x_1 \neq x$, and so $x$ is a limit point of $T$ *i.e.*, $x \in T'$.

$\therefore \quad x \in S' \Rightarrow x \in T'$. Hence $S' \subset T'$.

(*b*) We know $S \subset S \cup T \Rightarrow S' \subset (S \cup T)'$, by part (*a*).

Similarly, $T \subset S \cup T \Rightarrow T' \subset (S \cup T)'$.

$\therefore \quad S' \cup T' \subset (S \cup T)'$. ...(1)

Now we prove $(S \cup T)' \subset S' \cup T'$. ...(2)

Let $x \in (S \cup T)' \Rightarrow x$ is a limit point of $S \cup T$

$\Rightarrow$ every *nbd.* of $x$ contains a point $y \in S \cup T$, $y \neq x$

$\Rightarrow$ every *nbd.* of $x$ contains a point ($y \in S$ or $y \in T$), $y \neq x$

$\Rightarrow$ every *nbd.* of $x$ contains a point $y \in S$, $y \neq x$,

or every *nbd.* of $x$ contains a point $y \in T$, $y \neq x$.

$\Rightarrow x$ is a limit point of $S$ or $x$ is a limit point of $T$

$\Rightarrow x \in S'$ or $x \in T'$

$\Rightarrow x \in S' \cup T'$. Thus (2) is proved.

From (1) and (2), $(S \cup T)' = S' \cup T'$.

(*c*) We know $S \cap T \subset S \Rightarrow (S \cap T)' \subset S'$, by part (*a*)

and $S \cap T \subset T \Rightarrow (S \cap T)' \subset T'$.

$\therefore \quad (S \cap T)' \subset S' \cap T'$.

**To prove that the equality does not hold, we take**

$S = ]1, 2[$ **and** $T = ]2, 3[$ **so that** $S \cap T = \phi$.

$\Rightarrow \quad S' = [1, 2], T' = [2, 3]$ **and**

$(S \cap T)' = \phi' = \phi$.

$S' \cap T' = \{2\}$ **and** $(S \cap T)' = \phi$.

$S \cap T)' \neq S' \cap T'$.

**Remark.** From Example 26 (*a*), it follows that

**If $x$ is a limit point of a set $S$ and $S \subset T$, then $x$ is also a limit point of the set $T$.**

**Example 27.** *If* $S, T \subseteq \mathbf{R}$, *show that* $x \in \mathbf{R}$ *is a limit point of* $S \cup T$ *if and only if* $x$ *is a limit point of* $S$ *or* $x$ *is a limit point of* $T$.

[D.U., B.A. (P) 1999]

**Hint.** Prove $(S \cup T)' = S' \cup T'$.

**Example 28.** *Show that if* $x \in \mathbf{R}$ *is a limit point of* $S \cap T$, *then* $x$ *is a limit point of* $S$ *and* $x$ *is a limit point of* $T$. *Is the converse true ? Justify your answer ?* [D.U., B.A. (P) 1999]

**Hint.** Prove part (*c*) of Example 26.

## 2.5 Some Theorems on Limit Points

**Theorem 4.** *A point* $p$ *is a limit point of a set* $A$ *if and only if every neighbourhood of* $p$ *contains infinitely many points of* $A$.

[D.U., B.A. (P) 1996 ; Maths (H) 1994]

**Proof.** *The condition is necessary.*

Let $p$ be a limit point of $A$. We shall show that every *nbd.* of $p$ contains infinitely many points of $A$. Suppose this is false. Then there exists *some* $\varepsilon > 0$ such that $]p-\varepsilon, p+\varepsilon[$ contains only a finite number of points, say $p_1, p_2, \ldots, p_n$ ; which are different from $p$.

Let $\delta = \min \{ |p-p_1|, |p-p_2|, \ldots, |p-p_n| \} \; (\delta > 0)$

$\Rightarrow \quad \delta \le |p-p_i|$ for $i = 1, 2, \ldots, n$

$\Rightarrow \quad ]p-\delta, p+\delta[$ contains no point of $A$ other than $p$

$\Rightarrow \quad p$ is not a limit point of $A$, which is a contradiction.

Hence every *nbd.* of $p$ must contain infinitely many points of $A$.

*The condition is sufficient.*

Suppose every *nbd.* of $p$ contains infinitely many points of $A$. Surely then, every *nbd.* of $p$ contains a point of $A$ different from $p$ and so $p$ is a limit point of $A$.

**Corollary.** *Show that a finite set has no limit point.*

[D.U., Maths (H) 1994]

**Proof.** Let $S$ be a finite set. Let, if possible, $p$ be a limit point of $S$. By Theorem 4, for every $\varepsilon > 0$, $]p-\varepsilon, p+\varepsilon[$ must contain an infinite number of elements of $S$, which is impossible, as $S$ is a finite set. Hence $p$ cannot be a limit point of $S$.

**Theorem 5. (Bolzano-Weierstrass Theorem)**

*Every infinite bounded set of real numbers has a limit point.*

[D.U., Maths (H) 1998, 94, 93, 92 ; B.A. (P) 1995]

**Proof.** Let $S$ be any infinite bounded set of real numbers with bounds $m = \inf S$ and $M = \sup S$ *i.e.*, $m \le x \le M \;\; \forall \; x \in S$. ...(1)

**Ex. 2.** *State Bolzano-Weierstrass Theorem. Give an example of an infinite bounded set which has a limit point.*

**Hint.** $S = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$ is an infinite bounded set having 0 as its limit point.

**Ex. 3.** *Verify Bolzano-Weierstrass theorem for the set*

$$S = \left\{ (-1)^n \frac{n}{n+1} : n \in \mathbf{N} \right\}.$$

**Hint.** Indeed $\quad S = \left\{ -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}, \dots \right\}$.

$S$ is an infinite bounded set, since $-1 < x < 1 \;\forall\; x \in S$.

Clearly, 0 is the limit point of $S$.

**Ex. 4.** *State Bolzano-Weierstrass theorem. Prove that the set*

$$\left\{ 3^n + \frac{1}{3^n} : n \in \mathbf{N} \right\}$$

*has no limit point. Does it contradict Bolzano-Weierstrass theorem ?*

[D.U., B.A. (P) 1999]

**Hint.** Refer to Example 22 (*iv*). The given set is infinite, unbounded and has no limit point and so it does not contradict Bolzano-Weierstrass theorem. Notice that Bolzano-Weierstrass theorem is contradicted, if an infinite bounded set has no limit point.

We have seen in Section 2.3 that *a set S is closed iff its complement* **R** ~ *S is an open set.* The following theorem characterises a closed set in terms of its limit points.

**Theorem 6.** *A set S is closed if and only if it contains all its limit points. Equivalently, S is closed* $\Leftrightarrow S' \subset S$.

[D.U., B.Sc. (G) 1998, 96 ; Maths (H) 1996, 95]

**Proof.** By definition of a closed set,

$S$ is closed $\Leftrightarrow$ $\mathbf{R} \sim S$ is an open set

$\Leftrightarrow$ $\mathbf{R} \sim S$ is a *nbd.* of each of its points

$\Leftrightarrow$ for each $x \in \mathbf{R} \sim S$, $\exists$ some $\varepsilon > 0$, such that

$]x - \varepsilon, x + \varepsilon[ \;\subset \mathbf{R} \sim S$ or $]x - \varepsilon, x + \varepsilon[ \;\cap S = \phi$

$(\because\; y \in \;]x - \varepsilon, x + \varepsilon[ \;\Rightarrow\; y \in \mathbf{R} \sim S \;\Rightarrow\; y \notin S)$

$\Leftrightarrow$ for each $x \in \mathbf{R} \sim S$, $\exists$ a *nbd.* of $x$ which contains no point of $S$

$\Leftrightarrow$ $x$ is not a limit point of $S$ $\;\forall\; x \in \mathbf{R} \sim S$

$\Leftrightarrow$ $x \notin S' \;\forall\; x \in \mathbf{R} \sim S$

$\Leftrightarrow$ $x \notin S' \;\forall\; x \notin S$

$\Leftrightarrow$ $S' \subset S$.

**Remark.** The above Theorem gives us a second definition of a closed

Clearly, $t < s < s + \varepsilon$. ...(2)

From (1) and (2), it follows that $]s - \varepsilon, s + \varepsilon[$ contains a point $t \in S$, $t \neq s$ for all $\varepsilon > 0$. Thus $s$ is a limit point of $S$. Since $S$ is a closed set, $s \in S$. Hence $s$ is a maximum of $S$. Similarly, $s' = \inf S$ is a minimum of $S$.

**Theorem 9.** *Prove that the derived set of any set is a closed set.*

**Proof.** Let $S'$ be the derived set of a set S. We have to show that $S'$ is a closed set. Equivalently, we have to show that if $p$ is a limit point of $S'$, then $p \in S'$. (See Remark of Theorem 6)

Suppose $p$ is a limit point of $S'$. Then for any $\varepsilon > 0$, $]p - \varepsilon, p + \varepsilon[$ contains a point $q$ of $S'$, $q \neq p$. Now $q \in S' \Rightarrow q$ is a limit point of $S$.

$\Rightarrow$ Every *nbd.* of $q$ (in particular, $]p - \varepsilon, p + \varepsilon[$ ) contains a point $r$ of $S$, $r \neq q$

$\Rightarrow$ for each $\varepsilon > 0$, $]p - \varepsilon, p + \varepsilon[$ contains a point $r \in S$, where $r \neq p$ $(\because r \neq q$ and $q \neq p)$

$\Rightarrow p$ is a limit point of $S \Rightarrow p \in S'$. Hence $S'$ is closed set.

**Theorem 10.** *The derived set of a bounded set is bounded.*

**Proof.** Let $S$ be any bounded set whose bounds are

$$m = \inf S \quad \text{and} \quad M = \sup S \quad \textit{i.e.,} \quad m \leq x \leq M \quad \forall \quad x \in S. \quad \ldots(1)$$

We shall prove that $m \leq x \leq M \quad \forall \quad x \in S'$.

Let, if possible, there exist some $y \in S'$ such that $y > M$.

Now $y > M \quad \Rightarrow \quad y - M > 0$. We choose $\varepsilon > 0$ such that

$$\varepsilon < y - M \quad \Rightarrow \quad M < y - \varepsilon. \quad \ldots(2)$$

From (1) and (2), $\quad x \leq M < y - \varepsilon \quad \forall \quad x \in S$.

It follows that there exists a *nbd.* $]y - \varepsilon, y + \varepsilon[$ of $y$ which does not contain any point of $S$ and so $y$ is not a limit point of $S$ *i.e.,* $y \notin S'$, which is a contradiction.

Thus $\quad x \leq M \quad \forall \quad x \in S'$.

Similarly, we can show that $\quad m \leq x \quad \forall \quad x \in S'$.

Hence $m \leq x \leq M \quad \forall \quad x \in S'$ and so $S'$ is bounded.

**Corollary.** *Let S be a bounded infinite non-empty subset of* **R**. *Prove that the derived set of S has the smallest and the greatest members.*

[D.U., B.Sc. (G) 1999]

**Proof.** By Theorems 9 and 10, derived set $S'$ of $S$ is closed and bounded. Since $S$ is an infinite bounded set, $S$ has a limit point (B-W Theorem) and so $S'$ is non-empty. Hence $S'$ is a non-empty, closed and bounded subset of **R** and so $S'$ has the greatest and smallest members (by Theorem 8).

## EXAMPLES

**Example 29.** *Show that* $S = \{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots\}$ *is closed*

**Example 34.** *Find the limit points of the following :*

*(i)* **N**. *(ii)* $]a, b]$. *(iii)* $\mathbf{R} \sim \mathbf{Q}$. *(iv) finite set.*

**Solution.** *(i)* **N** has no limit point.

*(ii)* $\{]a, b]\}' = [a, b]$. Hence each point of $[a, b]$ is a limit point of $]a, b]$.

*(iii)* $\mathbf{R} \sim \mathbf{Q}$ consists of all irrational numbers. Every real number $p \in \mathbf{R}$ is a limit point of $\mathbf{R} \sim \mathbf{Q}$, since for each $\varepsilon > 0$, $]p - \varepsilon, p + \varepsilon[$ contains an irrational number other than $p$.

*(iv)* A finite set has no limit point (Cor. 1 of Theorem 6).

**Example 35.** *Show that the set* $S = \left\{ \frac{1 + (-1)^n}{n} : n \in \mathbf{N} \right\}$ *is closed.*

**Solution.** We have $S = \left\{ 0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \dots \right\}$ and $S' = \{0\}$.

Since $S' \subset S$, $S$ is a closed set.

**Example 36.** *State whether the following subsets of* **R** *are open, closed :*

*(i) the set* **Z** *of all integers.*

**Ans.** **Z** is closed. ($\because$ $\mathbf{Z}' = \phi \subset \mathbf{Z}$.)

However, **Z** is not open as **Z** is not a *nbd.* of any of its points.

*(ii)* $S = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\}$.

**Ans.** $S$ is neither closed nor open.

We have $S' = \{0\} \not\subset S$. Also $S$ is not a *nbd.* of any of its points.

*(iii) The segment* $]a, b[$.

**Ans.** It is an open set, since it is an open interval.

However, it is not a closed set, since $\{]a, b[\}' = [a, b] \not\subset\, ]a, b[$.

*(iv) The set* **N** *of natural numbers.*

**Ans.** **N** is closed. ($\because$ $\mathbf{N}' = \phi \subset \mathbf{N}$.)

However, **N** is not open as **N** is not a *nbd.* of any of its points.

*(v)* $S = \{1, 2, 3, 4\}$.

**Ans.** $S$ is a closed set, since it is a finite set and every finite set is closed. [See Cor. 1, Theorem 6]

However, $S$ is not open, since $S$ is not a *nbd.* of any of its points.

*(vi) The set* **R** *of real numbers.*

**Ans.** **R** is both open and closed.

(*vii*) $S = ]0, 1[ \cup ]2, 3[$.

**Ans.** $S$ being the union of two open sets is an open set.

Now $S' = \{]0, 1[\}' \cup \{]2, 3[\}' = [0, 1] \cup [2, 3] \not\subset S$. [See Example 26(*b*)]

Hence. $S$ is not a closed set.

(*viii*) $S = \left\{ 1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \dots \right\}$.

**Ans.** $S' = \{0\} \not\subset S$ and so $S$ is not closed.

Since $S$ is not a *nbd.* of each of its points, $S$ is not open.

(*ix*) *The set* $\mathbf{Q}$ *as rational numbers.*

**Ans.** We know $\mathbf{Q}' = \mathbf{R} \not\subset \mathbf{Q}$ and so $\mathbf{Q}$ is not closed.

Further $\mathbf{Q}$ is not a *nbd.* of any of its points and so $\mathbf{Q}$ is not open.

(*x*) $S = [0, 1] \cup [2, 3]$.

**Ans.** $S$ being the union of two closed sets is a closed set (Every closed interval is a closed set).

Since $S$ is not a *nbd.* of the points 0, 1, 2, and 3, $S$ is not an open set.

**Example 37.** *If F is a closed bounded set, then every infinite subset S of F has a limit point in F.*

**Solution.** Since $S$ is a subset of the bounded set $F$, so $S$ is a bounded and infinite set. By Bolzano-Weierstrass Theorem, $S$ has a limit point, say $p$. Since $S \subset F$, so $p$ is also a limit point of $F$. Since $F$ is a closed set and $p$ is a limit point of $F$, so $p \in F$. Hence $S$ has its limit point in $F$.

**Example 38.** *If A is open and B is closed, show that* $A \sim B$ *is open and* $B \sim A$ *is closed.*

**Solution.** Since $B$ is closed, $\mathbf{R} \sim B$ is an open set. Now $A \sim B = A \cap (\mathbf{R} \sim B)$, which being the intersection of two open sets is an open set. Thus $A \sim B$ is an open set.

Again $B \sim A = B \cap (\mathbf{R} \sim A)$, which being the intersection of two closed sets is a closed set. Thus $B \sim A$ is a closed set.

---

# 3

# Sequences

## 3.1 Sequence

**Definition.** *A* **sequence** *of real numbers (or a* **real sequence***) is defined as a function* $f : \mathbf{N} \to \mathbf{R}$, where $\mathbf{N}$ is the set of natural numbers and $\mathbf{R}$ is the set of real numbers.

If $f : \mathbf{N} \to \mathbf{R}$ is a sequence then for each $n \in \mathbf{N}$, $f(n)$ is a real number. It is customary to write $f(n)$ as $f_n$. A sequence may be written as

$$\langle f_1, f_2, f_3, \dots, f_n, \dots \rangle \quad \text{or} \quad \langle f_n \rangle \quad \text{or} \quad \{ f_n \}.$$

The real numbers $f_1, f_2, \dots, f_n, \dots$ are called the *terms* or *elements* of the sequence. $f_1$ is called the first term, $f_2$ the second term, ..., $f_n$ the $n$th term of the sequence $\langle f_n \rangle$. We can write all the terms of a sequence if its $n$th term is known.

**Illustrations**

1. $\langle n^2 \rangle = \langle 1, 4, 9, 16, \dots \rangle$.
2. $\langle \frac{1}{n} \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \rangle$.
3. $\langle (-1)^n \rangle = \langle -1, 1, -1, 1, \dots \rangle$.
4. $\langle 1 + (-1)^n \rangle = \langle 0, 2, 0, 2, \dots \rangle$.
5. A sequence $\langle a_n \rangle$ may be defined by a *recursion formula* :

$$a_{n+1} = \sqrt{2a_n}, \; a_1 = 1.$$

Here the terms of the sequence are $1, \sqrt{2}, \sqrt{2\sqrt{2}}, \dots$

We notice that the terms of a sequence may be all distinct (as in **1**, **2** and **5**) or repeating (as in **3** and **4**).

*The set of all the distinct elements of a sequence is called the* **range set** *of the given sequence.*

The range sets of the sequences given in **1** to **5** are, respectively

(*i*) $\{1, 4, 9, \dots\}$, (*ii*) $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ (*iii*) $\{-1, 1\}$,

(*iv*) $\{0, 2\}$ (*v*) $\{1, \sqrt{2}, \sqrt{2\sqrt{2}}, \dots\}$.

Thus it follows that *the range set of a sequence may be finite or infinite but the sequence has always an infinite number of elements.*

## 3.2 Bounds of a Sequence

**Definition 1.** *A sequence* $\langle a_n \rangle$ *is said to be* **bounded above**, *if there exists a real number K such that* $a_n \le K \;\; \forall \, n \in \mathbf{N}$.

*K is called a* ~~**lower**~~ upper **bound** *of the sequence* $\langle a_n \rangle$.

With this motivation, we are now in a position to give the proper definition of a *convergent sequence*.

**Definition.** *A sequence* $\langle a_n \rangle$ *is said to* **converge** *to a number* $l$, *if given any* $\varepsilon > 0$, *there exists a positive integer* $m$ *(depending on* $\varepsilon$*), such that*

$$|a_n - l| < \varepsilon \quad \forall \quad n \geq m.$$

*The number* $l$ *is called the* **limit** *of the sequence* $\langle a_n \rangle$ *and is written as*

$$\lim_{n \to \infty} a_n = l \quad \text{or} \quad \lim a_n = l \quad \text{or} \quad a_n \to l.$$

## EXAMPLES

**Example 1.** *Show that* $\lim\limits_{n \to \infty} \frac{1}{n} = 0.$

**Solution.** Let $\varepsilon > 0$ be any number. If $a_n = \frac{1}{n}$, then

$$|a_n - 0| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \varepsilon, \text{ for } n > \frac{1}{\varepsilon}.$$

Let $m$ be a positive integer such that $m > 1/\varepsilon$. Then

$|a_n - 0| < \varepsilon \ \forall \ n \geq m$. Hence $\lim\limits_{n \to \infty} a_n = 0.$

**Example 2.** *Show that the constant sequence* $\langle 1, 1, 1, \ldots \rangle$ *converges to* 1.

**Solution.** We have $a_n = 1 \quad \forall \quad n.$

$\therefore \qquad |a_n - 1| = 0 \quad \forall \quad n \geq 1.$

Let $\varepsilon > 0$ be given. Then $|a_n - 1| < \varepsilon \ \forall \ n \geq m$, where $m = 1$.

Hence the given sequence converges to 1.

**Example 3.** *Show that* $\lim\limits_{n \to \infty} a_n = 1$, *where* $a_n = 1 + \frac{(-1)^n}{n}$.

**Solution.** Let $\varepsilon > 0$ be any number. Then

$$|a_n - 1| = \left| 1 + \frac{(-1)^n}{n} - 1 \right| = \frac{1}{n} < \varepsilon, \text{ if } n > \frac{1}{\varepsilon}.$$

$\left|\frac{(-1)^n}{n}\right| = \frac{1^n}{n}$

Let $m$ be a positive integer $> 1/\varepsilon$. Then

$|a_n - 1| < \varepsilon \quad \forall \quad n \geq m.$ Hence $\lim\limits_{n \to \infty} a_n = 1.$

**Example 4.** *Show that* $\lim\limits_{n \to \infty} \frac{1 + 2 + 3 + \ldots + n}{n^2} = \frac{1}{2}.$

**Solution.** Let $a_n = \frac{1 + 2 + 3 + \ldots + n}{n^2} = \frac{\frac{1}{2} n (n + 1)}{n^2} = \frac{n + 1}{2n}.$

**Solution.** We have $|a_n - 2| = \left| -\frac{1}{2^n} \right| = \frac{1}{2^n}$.

Let $\varepsilon > 0$ be given. Then $|a_n - 2| < \varepsilon$, if $\frac{1}{2^n} < \varepsilon$ or $2^n > \frac{1}{\varepsilon}$.

Now $2^n > 1/\varepsilon \Rightarrow n \log 2 > \log 1/\varepsilon \Rightarrow n > (\log 1/\varepsilon)/\log 2$.
Let $m$ be a positive integer $\geq (\log 1/\varepsilon)/\log 2$. Then

$$|a_n - 2| < \varepsilon \text{ for } n \geq m. \text{ Hence } \lim_{n \to \infty} a_n = 2.$$

**Example 8.** *Prove that if $p > 0$, then $\lim_{n \to \infty} \frac{1}{n^p} = 0$.*

**Solution.** Let $\varepsilon > 0$ be given. We have

$$\left| \frac{1}{n^p} - 0 \right| = \frac{1}{n^p} < \varepsilon \Rightarrow n^p > \frac{1}{\varepsilon} \Rightarrow n > \left( \frac{1}{\varepsilon} \right)^{1/p} \Rightarrow n \geq m,$$

where $m$ is a positive integer $\geq (1/\varepsilon)^{1/p}$.

Thus $\left| \frac{1}{n^p} - 0 \right| < \varepsilon$ for $n \geq m$ and so $\lim_{n \to \infty} \frac{1}{n^p} = 0$.

4 **Example 9.** *Show that the sequence $\langle r^n \rangle$ converges to zero if $|r| < 1$.* [D.U., Physics (H) 1993]

*Or*

*Show that $\lim_{n \to \infty} r^n = 0$, if $|r| < 1$.*

**Solution.** Since $|r| < 1$, so there exists some $h > 0$ such that

$$|r| = \frac{1}{1+h} \Rightarrow |r|^n = \frac{1}{(1+h)^n}. \qquad \text{...(1)}$$

Now $\quad (1+h)^n = 1 + nh + \frac{n(n-1)}{2} h^2 + \ldots + h^n$ (Binomial Theorem)

or $\quad (1+h)^n \geq 1 + nh \quad \forall\, n \qquad (\because h > 0)$

or $$\frac{1}{(1+h)^n} \leq \frac{1}{1+nh} \quad \forall\, n. \qquad \text{...(2)}$$

From (1) and (2), we obtain

$$|r^n - 0| = |r^n| = |r|^n = \frac{1}{(1+h)^n} \leq \frac{1}{1+nh} \quad \forall\, n.$$

Let $\varepsilon > 0$ be given. Then

$$|r^n - 0| \leq \frac{1}{1+nh} < \varepsilon, \quad \text{if} \quad 1 + nh > \frac{1}{\varepsilon}$$

or $$|r^n - 0| < \varepsilon, \quad \text{if} \quad n > \frac{1}{h}\left(\frac{1}{\varepsilon} - 1\right).$$

Let $m$ be a positive integer $> \frac{1}{h}\left(\frac{1}{\varepsilon} - 1\right)$. Then

$$|r^n - 0| < \varepsilon \quad \forall \quad n \geq m.$$

Hence $\lim\limits_{n \to \infty} r^n = 0$, provided $|r| < 1$.

**Remark.** As an application of the above result, we see that

$$\lim_{n \to \infty} \frac{1}{2^n} = 0, \quad \lim_{n \to \infty} \frac{1}{3^n} = 0 \text{ etc.} \qquad \left(\because \frac{1}{2} < 1, \frac{1}{3} < 1\right)$$

✓**Example 10.** *Show that* $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1.$

[D.U., Maths (H) 1997 ; B.Sc. (G) 1993 ; B.A. (P) 1999, 96]

**Solution.** Let $\sqrt[n]{n} = n^{1/n} = 1 + h$, where $h \geq 0$

$$\Rightarrow n = (1+h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + \ldots + h^n \quad \text{(Binomial Theorem)}$$

$$\Rightarrow n > \frac{n(n-1)}{2}h^2, \quad \forall \quad n \geq 2 \quad [\because h \geq 0]$$

$$\Rightarrow \quad h^2 < \frac{2}{n-1}, \quad \forall \quad n \geq 2$$

$$\Rightarrow \quad |h| < \sqrt{\frac{2}{n-1}}, \quad \forall \quad n \geq 2.$$

Let $\varepsilon > 0$ be given. Then $|1+h-1| = |h| < \varepsilon$

$$|h| < \sqrt{\frac{2}{n-1}} < \varepsilon, \quad \text{if} \quad \frac{2}{n-1} < \varepsilon^2 \quad \text{or} \quad n > 1 + \frac{2}{\varepsilon^2}.$$

Let $m$ be a positive integer $> 1 + \frac{2}{\varepsilon^2}$. Then

$$|\sqrt[n]{n} - 1| = |h| < \varepsilon \quad \text{for} \quad n \geq m. \qquad \text{Hence} \quad \lim_{n \to \infty} \sqrt[n]{n} = 1.$$

## EXERCISES

1. Show that sequence $\langle 1 + \frac{1}{n} \rangle$ converges to 1.
2. Show that a constant sequence $\langle c, c, c, \ldots \rangle$ converges to $c$.
   [**Hint.** If $\varepsilon > 0$, then $|c_n - c| = |c - c| = 0 < \varepsilon \ \forall \ n \geq 1$.]
3. Show that $\lim\limits_{n \to \infty} \frac{3n+2}{n+1} = 3.$
4. Show that $\lim\limits_{n \to \infty} \frac{n^2+1}{2n^2+5} = \frac{1}{2}.$

5. Show that $\lim\limits_{n \to \infty} \frac{1+3+5+\ldots+(2n-1)}{n^2} = 1.$
   [**Hint.** $1 + 3 + 5 + \ldots + (2n-1) = n^2$]
6. Show that $\lim\limits_{n \to \infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2.$
   $\left[\textbf{Hint.} \left|\frac{3+2\sqrt{n}}{\sqrt{n}} - 2\right| = \frac{3}{\sqrt{n}} < \varepsilon \text{ if } n \geq \frac{9}{\varepsilon^2}. \text{ Take } m \geq \frac{9}{\varepsilon^2}.\right]$
7. Show that $\langle 1 + (-1)^n \rangle$ is not convergent.
   [**Hint.** Similar to Example 6.]

### 3.4 Non-Convergent Sequences

**Definition 1.** *A sequence* $\langle a_n \rangle$ *is said to* **diverge to** $+\infty$, *denoted as* $\lim\limits_{n \to \infty} a_n = +\infty$, *if to each positive number G, however large, there exists a positive integer m such that*

$$a_n > G \quad \forall \quad n \geq m.$$

**Definition 2.** *A sequence* $\langle a_n \rangle$ *is said to* **diverge to** $-\infty$, *denoted as* $\lim\limits_{n \to \infty} a_n = -\infty$, *if to each negative number g, however small, there exists a positive integer m such that*

$$a_n < g \quad \forall \quad n \geq m.$$

**Definition 3.** *A sequence which diverges to either* $+\infty$ *or* $-\infty$ *is said to be a* **divergent** *sequence.*

**Definition 4.** *A sequence which diverges to neither* $+\infty$ *nor* $-\infty$ *is said to be a* **non-divergent** *sequence.*

**Illustrations**

1. $\langle n^2 \rangle = \langle 1, 4, 9, 16, \ldots \rangle$ diverges to $+\infty$.
2. $\langle 2^n \rangle = \langle 2, 4, 8, 16, 32, \ldots \rangle$ diverges to $+\infty$.
3. $\langle -2n \rangle = \langle -2, -4, -6, -8, \ldots \rangle$ diverges to $-\infty$.
4. $\langle -n^2 \rangle = \langle -1, -4, -9, -16, \ldots \rangle$ diverges to $-\infty$.
5. ~~$\langle (-1)^n n \rangle = \langle -1, 2, -3, 4, -5, 6, \ldots \rangle$~~
   neither diverges to $+\infty$ nor $-\infty$.

**Definition 5.** *A sequence is said to* **oscillate finitely**, *if*

*(i) it is bounded,*

*(ii) it neither converges nor diverges.*

**Definition 6.** *A sequence is said to* **oscillate infinitely**, *if*

*(i) it is not bounded,*

*(ii) it neither converges nor diverges.*

Let $K = \max\{a_1, a_2, \ldots, a_{m-1}, l+1\}$. ...(*ii*)

$k = \min\{a_1, a_2, \ldots, a_{m-1}, l-1\}$. ...(*iii*)

From (*i*), (*ii*), (*iii*), we get

$$k \le a_n \le K \ \ \forall\, n.$$

Hence $\langle a_n \rangle$ is bounded.

**Remark.** *The converse of the theorem need not be true.*

For example $\langle (-1)^n = \langle -1, 1, -1, 1, \ldots \rangle \rangle$ is bounded, but is not convergent. [See Example 6]

## 3.6 Algebra of Sequences

**Definition.** Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be two sequences. Then

$$\langle a_n + b_n \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots \rangle,$$
$$\langle a_n - b_n \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots \rangle,$$
$$\langle a_n b_n \rangle = \langle a_1 b_1, a_2 b_2, a_3 b_3, \ldots \rangle,$$
$$\left\langle \frac{a_n}{b_n} \right\rangle = \left\langle \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots \right\rangle \qquad (b_n \ne 0 \ \forall\, n),$$

are respectively called the **sum, difference, product** and **quotient** of the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

**Theorem 3.** *If* $\langle a_n \rangle, \langle b_n \rangle$ *be two sequences such that*

$$\lim a_n = a, \ \ \lim b_n = b, \text{ then}$$

(*i*) $\lim (a_n \pm b_n) = \lim a_n \pm \lim b_n = a \pm b.$

(*ii*) $\lim (a_n b_n) = (\lim a_n)(\lim b_n) = ab.$ [D.U., B.Sc. (G) 1993]

(*iii*) $\lim (a_n/b_n) = (\lim a_n/\lim b_n) = a/b.$ $(b \ne 0, b_n \ne 0 \ \forall\, n)$ [D.U., B.Sc. (G) 1990]

**Proof.** Let $\varepsilon > 0$ be given.

(*i*) Since $\lim a_n = a$ and $\lim b_n = b$, therefore, there exist positive integers $m_1$ and $m_2$ such that

$$|a_n - a| < \frac{1}{2}\varepsilon \ \ \forall\, n \ge m_1, \quad |b_n - b| < \frac{1}{2}\varepsilon \ \ \forall\, n \ge m_2.$$

Let $m = \max(m_1, m_2)$. Then

$$|a_n - a| < \frac{1}{2}\varepsilon, \ |b_n - b| < \frac{1}{2}\varepsilon, \ \ \forall \ \ n \ge m. \qquad \ldots(1)$$

Now
$$\begin{aligned} |(a_n \pm b_n) - (a \pm b)| &= |(a_n - a) \pm (b_n - b)| \\ &\le |a_n - a| + |b_n - b| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \ \ \forall\ n \ge m, \text{ using (1)} \end{aligned}$$

Hence $\lim (a_n \pm b_n) = a \pm b = \lim a_n \pm \lim b_n.$

(*ii*) The sequence $\langle a_n \rangle$ **being convergent is bounded and so there exists** umber $k$ such that $|a_n| \le k \ \forall\, n.$

Since $\lim a_n = a$, $\lim b_n = b$, therefore, there exist positive integers $m_2, m_3$ such that

$$|a_n - a| < \frac{1}{4}|b|\varepsilon, \quad \forall \quad n \geq m_2 \qquad \text{...(8)}$$

and
$$|b_n - b| < \frac{1}{4}\frac{|b|^2 \varepsilon}{|a|+1}, \forall \quad n \geq m_3. \qquad \text{...(9)}$$

Let $m = \max(m_1, m_2, m_3)$. Then, from (7), (8) and (9), we have

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \frac{2}{|b|}\left(\frac{1}{4}|b|\varepsilon\right) + \frac{2|a|}{|b|^2}\left(\frac{1}{4}\cdot\frac{|b|^2\varepsilon}{|a|+1}\right), \forall n \geq m$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \forall \ n \geq m. \qquad \left(\because \frac{|a|}{|a|+1} < 1\right)$$

Hence $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b} = \frac{\lim a_n}{\lim b_n}$

**Ex.** (*i*) *If* $\lim a_n = a$ *show that* $\lim (ka_n) = ka$, *k being a fixed real number.*

(*ii*) *If* $\lim a_n = a$, *where* $a \neq 0$ *and* $a_n \neq 0 \ \forall \ n \in \mathbf{N}$, *show that* $\lim (1/a_n) = 1/a$.

**Hint.** (*i*) $\lim (k a_n) = \lim k . \lim a_n = k a$.

(*ii*) $\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a - a_n|}{|a||a_n|} < \frac{|a_n - a|}{2|a|^2} \ \forall n \geq m_1$, by eqn. (6) of Theorem 3.

Now $\lim a_n = a \Rightarrow |a_n - a| < 2\varepsilon|a|^2 \ \forall \ n \geq m_2$.

If $m = \max(m_1, m_2)$, then $\left|\frac{1}{a_n} - \frac{1}{a}\right| < \varepsilon \ \forall \ n \geq m$.

## EXAMPLES

**Example 11.** *Show that* $\lim\limits_{n \to \infty} \frac{1}{n^2} = 0$.

**Solution.** We know $\lim\limits_{n \to \infty} \frac{1}{n} = 0$. By Theorem 3, we have

$$\lim_{n \to \infty} \frac{1}{n^2} = \lim_{n \to \infty}\left(\frac{1}{n}\cdot\frac{1}{n}\right) = \lim_{n \to \infty}\frac{1}{n} . \lim_{n \to \infty}\frac{1}{n} = 0.$$

**Example 12.** *Show that* $\lim\limits_{n \to \infty} \frac{2n^2 + 3}{3n^2 + 5n} = \frac{2}{3}$.

**Solution.** Dividing numerator and denominator by $n^2$, we get

$$\frac{2n^2 + 3}{3n^2 + 5n} = \frac{2 + (3/n^2)}{3 + (5/n)}.$$

$$\therefore \quad \lim_{n \to \infty} \frac{2n^2 + 3}{3n^2 + 5n} = \frac{\lim\{2 + (3/n^2)\}}{\lim\{3 + (5/n)\}} = \frac{2 + 3\lim(1/n^2)}{3 + 5\lim(1/n)}$$

Then $\langle a_n \rangle$ does not converge to any limit.

But $\quad \langle |a_n| \rangle = \langle 1, 1, 1 \ldots \rangle$ converges to 1.

Thus $\langle |a_n| \rangle \to |1| = 1$, but $\langle a_n \rangle$ does not converge to 1.

**Example 16.** *Prove that $a_n \to 0$ if and only if $|a_n| \to 0$.*

**Solution.** Let $\varepsilon > 0$ be given. We have

$$|a_n - 0| = |a_n| = |\,|a_n| - 0|$$

$$\therefore \qquad |a_n - 0| < \varepsilon \Leftrightarrow |\,|a_n| - 0| < \varepsilon \quad \forall \quad n \geq m.$$

Hence $\qquad a_n \to 0 \Leftrightarrow |a_n| \to 0.$

**Note.** It may be observed that the converse of Example 15 is true if $a = 0$.

$\left|\frac{\sin n\pi}{n}\right| = \left|\frac{1}{n}\right| |\sin n\pi|$

**Example 17.** *Show that* $\lim\limits_{n \to \infty} \dfrac{\sin(n\pi)}{n} = 0.$

**Solution.** We know $\left|\dfrac{\sin(n\pi)}{n}\right| \leq \left|\dfrac{1}{n}\right| \to 0$ as $n \to \infty$

$$\Rightarrow \qquad \left|\frac{\sin(n\pi)}{n}\right| \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \qquad \frac{\sin(n\pi)}{n} \to 0 \text{ as } n \to \infty, \text{ by Example 16.}$$

**Example 18.** *Show that if $\langle a_n \rangle$ converges and $b_n \to +\infty$, then $a_n + b_n \to +\infty$.*

**Solution.** Let $\lim\limits_{n \to \infty} a_n = l$. We have $\lim\limits_{n \to \infty} b_n = +\infty$.

Now $\qquad \lim\limits_{n \to \infty} (a_n + b_n) = \lim\limits_{n \to \infty} a_n + \lim\limits_{n \to \infty} b_n = l + \infty = +\infty.$

Hence $\quad a_n + b_n \to +\infty.$

**Example 19.** *If $\langle a_n \rangle$ converges and $\langle b_n \rangle$ diverges, show that (i) $\lim (a_n/b_n) = 0$, (ii) $\langle a_n + b_n \rangle$ is divergent.*

**Solution.** Let $\lim\limits_{n \to \infty} a_n = l$ and $\lim\limits_{n \to \infty} b_n = +\infty$.

Then
$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n \to \infty} a_n}{\lim\limits_{n \to \infty} b_n} = \frac{l}{+\infty} = 0,$$

and
$$\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = l + \infty = +\infty,$$

Hence $\langle a_n + b_n \rangle$ is divergent.

**Example 20.** *(i) If $a_n \to +\infty$ and $b_n \geq a_n$ for all $n$, show that $b_n \to +\infty$.*

*(ii) If $a_n \to -\infty$ and $b_n \leq a_n$ for all $n$, show that $b_n \to -\infty$.*

**Solution.** (*i*) Since $a_n \to +\infty$, therefore, given any number $G > 0$, however large, there exists a positive integer $m$ such that

$$a_n > G \quad \forall\ n \geq m. \qquad \text{...(1)}$$

We have $\quad b_n \geq a_n \quad \forall\ n. \qquad \text{...(2)}$

From (1) and (2), $b_n > G \quad \forall \quad n \geq m.$ Hence $b_n \to +\infty$.

(*ii*) Since $a_n \to -\infty$, for any negative number $g$, however small, there exists a positive integer $m$ such that

$$a_n < g \quad \forall \quad n \geq m. \qquad \text{...(3)}$$

We have
$$b_n \leq a_n \quad \forall \quad n. \qquad \text{...(4)}$$

From (3) and (4), $\quad b_n < g \quad \forall \quad n \geq m.$ Hence $b_n \to -\infty$.

**Example 21.** *Let* $\langle a_n \rangle$ *and* $\langle b_n \rangle$ *be two sequences such that* $\langle a_n + b_n \rangle$ *and* $\langle a_n b_n \rangle$ *converge. Show, by an example, that the sequences* $\langle a_n \rangle$ *and* $\langle b_n \rangle$ *may fail to converge.*

**Solution.** Let $\langle a_n \rangle = \langle (-1)^n \rangle = \langle -1, 1, -1, 1, \ldots \rangle$,

$$\langle b_n \rangle = \langle (-1)^{n+1} \rangle = \langle 1, -1, 1, -1, \ldots \rangle.$$

Then $\langle a_n \rangle$ and $\langle b_n \rangle$ both do not converge, but

$$\langle a_n + b_n \rangle = \langle 0, 0, 0 \ldots \rangle \quad \text{converges to } 0,$$

$$\langle a_n b_n \rangle = \langle -1, -1, -1 \ldots \rangle \text{ converges to } -1.$$

**Example 22.** *Let* $\langle a_n \rangle$ *and* $\langle b_n \rangle$ *be two sequences such that* $\langle a_n - b_n \rangle$ *and* $\langle a_n/b_n \rangle$ *converge. Show, by an example, that the sequences* $\langle a_n \rangle$ *and* $\langle b_n \rangle$ *may fail to converge.*

**Solution.** Let $\langle a_n \rangle = \langle b_n \rangle = \langle (-1)^n \rangle = \langle -1, 1, -1, 1 \ldots \rangle$.

Then $\langle a_n \rangle$ and $\langle b_n \rangle$ both do not converge, but

$$\langle a_n - b_n \rangle = \langle 0, 0, 0, \ldots \rangle \text{ converges to } 0,$$

$$\langle a_n/b_n \rangle = \langle 1, 1, 1, \ldots \rangle \text{ converges to } 1.$$

**Example 23.** *If* $a_n \to +\infty$ *and* $b_n \to -\infty$, *then show by examples that* $\langle a_n + b_n \rangle$ *may* (*i*) *converge,* (*ii*) *diverge to* $+\infty$, (*iii*) *diverge to* $-\infty$ *and* (*iv*) *oscillate.*

**Solution.** (*i*) Let $\quad \langle a_n \rangle = \langle n^2 \rangle \quad$ and $\quad \langle b_n \rangle = \langle -n^2 \rangle$.

Then $a_n \to +\infty$ and $b_n \to -\infty$, but

$$\langle a_n + b_n \rangle = \langle 0, 0, 0, \ldots \rangle \text{ converges to } 0.$$

(*ii*) Let $\quad \langle a_n \rangle = \langle 2n \rangle = \langle 2, 4, 6, 8, \ldots \rangle$,

$$\langle b_n \rangle = \langle -n \rangle = \langle -1, -2, -3, -4, \ldots \rangle.$$

Then $a_n \to +\infty$ and $b_n \to -\infty$, but

$$\langle a_n + b_n \rangle = \langle 1, 2, 3, 4, \ldots n, \ldots \rangle \quad \text{diverges to } +\infty.$$

(*iii*) Let $\quad \langle a_n \rangle = \langle n \rangle \quad$ and $\quad \langle b_n \rangle = \langle -2n \rangle$.

Then $a_n \to +\infty$ and $b_n \to -\infty$, but

$$\langle a_n + b_n \rangle = \langle -n \rangle \text{ diverges to } -\infty.$$

(*iv*) Let $\quad \langle a_n \rangle = \langle 1, 3, 9, 5, 25, 7, \ldots, n^2, n+1, \ldots \rangle$

$$\langle b_n \rangle = \langle -1, -2, -3, -4, -5, -6 \ldots, -n, \ldots \rangle.$$

Then $a_n \to +\infty \quad$ and $\quad b_n \to -\infty$, but

$$\langle a_n + b_n \rangle = \langle 0, 1, 6, 1, 20, 1, \ldots \rangle, \text{ which oscillates infinitely.}$$

**Example 24.** *Prove that if* $r > 1$, *then* $\lim\limits_{n \to \infty} r^n = +\infty$.

**Solution.** Since $r > 1$, we can take $r = 1 + h$, for some $h > 0$.

$$\therefore \quad r^n = (1+h)^n = 1 + nh + \tfrac{1}{2} n(n-1) h^2 + \ldots + h^n.$$

$$r^n > (1 + nh) \quad \forall \quad n. \qquad (\because\ h > 0) \qquad \text{...(1)}$$

If $G > 0$ be any number, however large, then

or $\quad 1 + nh > G \quad$ if $\quad n > \frac{1}{h}(G - 1)$.

Let $m$ be a positive integer $> \frac{1}{h}(G-1)$. Then

$$1 + nh > G \quad \forall \quad n \geq m. \qquad \text{...(2)}$$

From (1) and (2), $\quad r^n > G \quad \forall \quad n \geq m.$

Hence $\quad \lim\limits_{n \to \infty} r^n = +\infty$.

**Remark.** As an application of Example 24,

$$\lim_{n \to \infty} 2^n = \infty, \quad \lim_{n \to \infty} 3^n = \infty.$$

**Example 25.** *Given that* $\lim a_n = a$, $\lim b_n = b$, *and* $\langle S_n \rangle$ *and* $\langle T_n \rangle$ ***are two sequences, where*** $S_n = \max \{a_n, b_n\}$ *and* $T_n = \min \{a_n, b_n\}$. ***Show that the sequences*** $\langle S_n \rangle$ *and* $\langle T_n \rangle$ *are convergent and that*

$$\lim S_n = \max (a, b), \quad \lim T_n = \min (a, b).$$

**Solution.** For any two real numbers $x$ and $y$, we know

$$\max (x, y) = \frac{1}{2}(x + y) + \frac{1}{2}|x - y|, \qquad \text{...(1)}$$

$$\min (x, y) = \frac{1}{2}(x + y) - \frac{1}{2}|x - y|. \qquad \text{...(2)}$$

Taking $\quad x = a_n, \ y = b_n$ in (1), we obtain

$$S_n = \max (a_n, b_n) = \frac{1}{2}(a_n + b_n) + \frac{1}{2}|a_n - b_n|, \text{ using (1).}$$

$$\therefore \quad \lim S_n = \frac{1}{2} \lim (a_n + b_n) + \frac{1}{2} \lim |a_n - b_n|$$

$$= \frac{1}{2}(\lim a_n + \lim b_n) + \frac{1}{2} \lim |c_n|, \ c_n = a_n - b_n$$

$$= \frac{1}{2}(a+b) + \frac{1}{2}\lim|c_n|. \qquad \dots(3)$$

We have $\lim c_n = \lim (a_n - b_n) = \lim a_n - \lim b_n = a - b.$

$\therefore \quad \lim c_n = a - b \quad \Rightarrow \lim|c_n| = |a-b|$, by Example 15.

From (3), $\quad \lim S_n = \frac{1}{2}(a+b) + \frac{1}{2}|a-b| = \max(a, b)$

Similarly, $\quad \lim T_n = \min(a, b).$

**Example 26.** *Show that the sequence $\langle a_n \rangle$, where*

$$a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{n-1}}$$ *converges. Find* $\lim\limits_{n\to\infty} a_n$.

**Solution.** We have $a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{n-1}}$,

which is a geometric progression with common ratio $\frac{1}{3}$.

$$\therefore \quad a_n = \frac{\left(1 - \frac{1}{3^n}\right)}{\left(1 - \frac{1}{3}\right)} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right) = \frac{3}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^{n-1}$$

Now $\lim\limits_{n\to\infty} a_n = \frac{3}{2} - \frac{1}{2}\lim\limits_{n\to\infty}\left(\frac{1}{3}\right)^{n-1} = \frac{3}{2} - \frac{1}{2}\times 0 = \frac{3}{2}.$

$[\because \lim r^n = 0, \text{ if } |r| < 1].$

Hence $\langle a_n \rangle$ is convergent and $\lim\limits_{n\to\infty} a_n = \frac{3}{2}.$

**Example 27.** *Show that $\langle a_n \rangle$, $a_n = \sqrt{n}\,(\sqrt{n+1} - \sqrt{n})$, converges.*

**Solution.** We have

$$a_n = \frac{\sqrt{n}\,(\sqrt{n+1} - \sqrt{n})\,(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} \qquad \text{(Rationalize)}$$

or $$a_n = \frac{\sqrt{n}\,(n+1-n)}{\sqrt{n+1} + \sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}.$$

Hence $\lim\limits_{n\to\infty} a_n = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$ and so $\langle a_n \rangle$ converges.

**Example 28.** *Show that the sequence $\langle a_n \rangle$ defined by*

$$a_n = 0, \text{ if } n \text{ is odd}$$
$$= n, \text{ if } n \text{ is even}$$

*is not convergent.* **[B.C.A. Sample Problem]**

**Solution.** We have

$$\langle a_n \rangle = \langle 0, 2, 0, 4, 0, 6, 0, 8, \dots \rangle$$

Clearly, $\langle a_n \rangle$ is not bounded.

Hence $\langle a_n \rangle$ is not convergent, since every convergent sequence is necessarily bounded

**Example 29.** *Show that if $a_n \to a$, then $a_n^2 \to a^2$. Is the converse true ?* [D.U., Maths (H) 1999]

**Solution.** Since $\langle a_n \rangle$ is convergent, $\langle a_n \rangle$ is bounded and so there exists some $k > 0$ such that $|a_n| \le k \;\forall\; n \in \mathbf{N}$.

$$\begin{aligned}\text{Now} \quad |a_n^2 - a^2| &= |(a_n - a)(a_n + a)| = |a_n - a|\,|a_n + a| \\ &\le |a_n - a|\,\{|a_n| + |a|\} \\ &\le \{k + |a|\}\,|a_n - a| \;\forall\; n \in \mathbf{N}. \qquad \dots(1)\end{aligned}$$

Let $\varepsilon > 0$ be given. Since $a_n \to a$, there exists a positive integer $m$ such that

$$|a_n - a| < \frac{\varepsilon}{k + |a|} \quad \forall \; n \ge m. \qquad \dots(2)$$

From (1) and (2), we see that

$$|a_n^2 - a^2| < \varepsilon \quad \forall \; n \ge m.$$

Hence $\lim\limits_{n\to\infty} a_n^2 = a^2.$

(*ii*) However, the converse may not be true.

For example, let $\langle a_n \rangle = \langle (-1)^n \rangle$. Then

$$\langle a_n^2 \rangle = \langle 1, 1, 1, \dots \rangle \to 1 = 1^2,$$

but $\langle a_n \rangle$ does not converge to 1.

**Example 30.** *Show that if $a_n^2 \to 0$, then $a_n \to 0$ as $n \to \infty$.*

**Solution.** Let $\varepsilon > 0$ be arbitrary. Since $a_n^2 \to 0$, there exists a positive integer $m$ such that $|a_n^2 - 0| < \varepsilon^2 \;\forall\; n \ge m$

$\Rightarrow \quad |a_n^2| < \varepsilon^2 \;\forall\; n \ge m \;\Rightarrow\; a_n^2 < \varepsilon^2 \quad \forall\; n \ge m$

$\Rightarrow \quad |a_n| < |\varepsilon| \;\forall\; n \ge m \;\Rightarrow\; |a_n - 0| < \varepsilon \;\forall\; n \ge m.$

Hence $a_n \to 0$ as $n \to \infty$.

**Example 31.** *Let $\langle a_n \rangle$ be a sequence such that $\langle a_n^2 \rangle$ converges to zero. Is it necessary that $\langle a_n \rangle$ should converge to zero?*

**Solution.** Yes, it is necessary that $\lim\limits_{n\to\infty} a_n = 0.$ [See Example 30]

**Aliter.** Let, if possible, $\lim\limits_{n\to\infty} a_n = l \ne 0$, then

$$\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} a_n = l \cdot l = l^2 \neq 0,$$

which is contrary to the given hypothesis.

**Example 32.** *Show that if $a_n^2 \to a^2$, then $|a_n| \to |a|$ as $n \to \infty$.*

**Solution.** If $a = 0$, the result follows by Examples 30 and 16.

Let $a \neq 0$. Then

$$||a_n| - |a|| = \frac{||a_n| - |a|| \cdot ||a_n| + |a||}{||a_n| + |a||}$$

$$= \frac{||a_n|^2 - |a|^2|}{|a_n| + |a|} \le \frac{|a_n^2 - a^2|}{|a|}, \qquad \text{...(1)}$$

since $|a_n| + |a| \ge |a| \;\; \forall \;\; n \in \mathbf{N}$.

Let $\varepsilon > 0$ be given. Since $a_n^2 \to a^2$, there exists a positive integer $m$ such that

$$|a_n^2 - a^2| < \varepsilon |a| \qquad \forall \quad n \ge m. \qquad \text{...(2)}$$

From (1) and (2), we get

$$||a_n| - |a|| < \varepsilon \quad \forall \quad n \ge m.$$

Hence $\quad |a_n| \to |a| \;$ as $\; n \to \infty$.

**Example 33.** *Let $\langle a_n \rangle$ be a sequence of positive terms such that $a_n \to a$. Prove that $\sqrt{a_n} \to \sqrt{a}$.*

**Solution.** If $a = 0$, then $a_n \to 0$. For any $\varepsilon > 0$, there exists a positive integer $m$ such that

$$|a_n - 0| < \varepsilon \;\; \forall \; n \ge m$$

or $\qquad a_n < \varepsilon \;\forall\; n > m$, since $a_n > 0 \;\; \forall \;\; n$

Now $|\sqrt{a_n} - 0| = \sqrt{a_n} \le a_n < \varepsilon \;\forall\; n \ge m \Rightarrow \sqrt{a_n} \to 0$.

We now discuss the case when $a > 0$. We have

$$|\sqrt{a_n} - \sqrt{a}| = \left| \frac{(\sqrt{a_n} - \sqrt{a})(\sqrt{a_n} + \sqrt{a})}{\sqrt{a_n} + \sqrt{a}} \right|$$

$$\le \frac{|a_n - a|}{\sqrt{a}}, \text{ since } \sqrt{a_n} + \sqrt{a} \ge \sqrt{a} \;\; \forall \;\; n \in \mathbf{N}. \qquad \text{...(1)}$$

Since $a_n \to a$, there exists a positive integer $m_0$ such that

$$|a_n - a| < \varepsilon \sqrt{a} \;\; \forall \;\; n \ge m_0. \qquad \text{...(2)}$$

From (1) and (2), we obtain

$$|\sqrt{a_n} - \sqrt{a}| < \varepsilon \;\; \forall \;\; n \ge m_0.$$

Hence $\quad \lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}$.

**Example 34.** *Prove that if $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.*

[D.U., Physics (H) 1998]

**Solution.** If $p > 1$, let $\sqrt[n]{p} = 1 + h$, where $h > 0$.

$$\therefore \; p = (1 + h)^n = 1 + nh + \frac{n(n-1)}{2!} h^2 + .... + h^n.$$

$$> 1 + nh \quad \forall \quad n \;(\because\; h > 0).$$

Thus $\qquad \dfrac{p-1}{n} > h \quad$ or $\quad h < \dfrac{p-1}{n} \quad \forall \; n \in \mathbf{N}$.

Let $\varepsilon > 0$ be given. Then

$$|\sqrt[n]{p} - 1| = |h| = h < \frac{p-1}{n} < \varepsilon, \text{ if } n > \frac{p-1}{\varepsilon}.$$

Let $m$ be a positive integer $> (p-1)/\varepsilon$. Then

$$|\sqrt[n]{p} - 1| < \varepsilon \;\forall\; n \ge m \quad \Rightarrow \lim_{n\to\infty} \sqrt[n]{p} = 1.$$

Obviously, the result is true if $p = 1$.

Let $0 < p < 1$. Then $1/p > 1$ and as proved above,

$$\lim_{n\to\infty} \left(\frac{1}{p}\right)^{1/n} = 1 \qquad \text{or} \qquad \frac{1}{\lim\limits_{n\to\infty} (p)^{1/n}} = 1.$$

Hence $\quad \lim_{n\to\infty} \sqrt[n]{p} = 1$ for all $p > 0$.

**Example 35.** *If $a$ is a number greater than 1, prove that $\lim_{n\to\infty} \sqrt[n]{a} = 1$.*

[D.U., Physics (H) 1997]

*Is the result true if $0 < a \le 1$?*

**Solution.** Same as Example 34.

**Example 36.** *If $a_n > 0 \;\forall\; n \in \mathbf{N}$, prove that*

$$\lim a_n = 0 \;\; \Leftrightarrow \;\; \lim (1/a_n) = \infty.$$

**Solution.** Let $\lim a_n = 0$. Let $k > 0$ be arbitrary. For $\varepsilon = 1/k$, there exists a positive integer $m$ such that

$$|a_n - 0| < 1/k \;\;\forall\; n \ge m \;\Rightarrow\; a_n < 1/k \;\;\forall\; n \ge m \;\Rightarrow\; 1/a_n > k \;\forall\; n \ge m$$

Hence $\lim (1/a_n) = \infty$.

Conversely, $\lim (1/a_n) = \infty$ implies that for any $\varepsilon > 0$, there exists a positive integer $m_0$ such that

$$1/a_n > 1/\varepsilon \;\forall\; n \ge m_0 \;\Rightarrow 0 < a_n < \varepsilon \;\;\forall\; n \ge m \;\Rightarrow |a_n - 0| < \varepsilon \;\forall\; n \ge m.$$

Hence $\lim a_n = 0$.

Cauchy Convergence Criterion.

**Theorem 7. (Cauchy's first theorem on limits)**

*If* $\lim\limits_{n\to\infty} a_n = l$, *then* $\lim\limits_{n\to\infty} \dfrac{a_1 + a_2 + \dots + a_n}{n} = l.$

[D.U., B.Sc. (G) 1997, 95, 91; Physics (H) 1996, 93;
Maths (H) 1993, 91; B.A. (P) 1997, 90]

**Proof.** Let $b_n = a_n - l$ or $a_n = l + b_n \ \forall \ n \in \mathbf{N}$. ...(*i*)

Then $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} a_n - l = l - l = 0.$ ...(*ii*)

$\therefore \quad \lim\limits_{n\to\infty} b_n = 0.$

Now $\dfrac{a_1 + a_2 + \dots + a_n}{n} = l + \dfrac{b_1 + b_2 + \dots + b_n}{n}$, using (*i*)

To show that $\lim\limits_{n\to\infty} \left(\dfrac{a_1 + a_2 + \dots + a_n}{n}\right) = l$, we shall prove that

$$\lim_{n\to\infty} \left(\frac{b_1 + b_2 + \dots + b_n}{n}\right) = 0.$$

Let $\varepsilon > 0$ be any number. Now $\lim\limits_{n\to\infty} b_n = 0$ implies that there exists a positive integer $m$ such that

$$|b_n - 0| < \frac{\varepsilon}{2} \ \forall \ n \geq m \ \Rightarrow \ |b_n| < \frac{\varepsilon}{2} \ \forall \ n \geq m. \quad \text{...(iii)}$$

Again $\lim\limits_{n\to\infty} b_n = 0 \Rightarrow \langle b_n \rangle$ is convergent $\Rightarrow \langle b_n \rangle$ is bounded

$\Rightarrow$ there exists a number $K > 0$ such that $|b_n| \leq K \ \forall \ n \in \mathbf{N}$. ...(*iv*)

Now $\left|\dfrac{b_1 + b_2 + \dots + b_m}{n}\right| \leq \dfrac{1}{n}\left(|b_1| + |b_2| + \dots + |b_m|\right) \leq \dfrac{mK}{n}$, using (*iv*).

$\therefore \quad \left|\dfrac{b_1 + b_2 + \dots + b_m}{n}\right| < \dfrac{\varepsilon}{2}$, when $\dfrac{mK}{n} < \dfrac{\varepsilon}{2}$, *i.e.*, $n > \dfrac{2mK}{\varepsilon}$

Let $p$ be a positive integer $> \dfrac{2mK}{\varepsilon}$. Then

$$\left|\frac{b_1 + b_2 + \dots + b_m}{n}\right| < \frac{\varepsilon}{2} \ \forall \ n \geq p. \quad \text{...(v)}$$

Now $\left|\dfrac{b_{m+1} + b_{m+2} + \dots + b_n}{n}\right| \leq \dfrac{1}{n}\left(|b_{m+1}| + |b_{m+2}| + \dots + |b_n|\right)$

$$\leq \frac{1}{n}\underbrace{\left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \dots + \frac{\varepsilon}{2}\right)}_{(n-m)\text{ times}}, \text{ by (iii)}$$

$$= \left(\frac{n-m}{n}\right)\cdot\frac{\varepsilon}{2} < \frac{\varepsilon}{2}\ \forall\ n \ge m.$$

$\left(0\ \frac{n-m}{n} < 1\right)$

$$\therefore \quad \left|\frac{b_{m+1}+b_{m+2}+\ldots+b_n}{n}\right| < \frac{\varepsilon}{2}\quad \forall\ \ n \ge m. \qquad \ldots(vi)$$

$$\ldots(vii)$$

Let $q = \max(m, p)$. Then

$$\left|\frac{b_1+b_2+\ldots+b_n}{n}\right| = \left|\frac{b_1+b_2+\ldots+b_m}{n} + \frac{b_{m+1}+\ldots+b_n}{n}\right|$$

$$\le \left|\frac{b_1+b_2+\ldots+b_m}{n}\right| + \left|\frac{b_{m+1}+\ldots+b_n}{n}\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon\ \forall\ n \ge q,$$

using $(v)$, $(vi)$, $(vii)$

$$\therefore \quad \left|\frac{b_1+b_2+\ldots+b_n}{n}\right| = \left|\frac{b_1+b_2+\ldots+b_n}{n} - 0\right| < \varepsilon\quad \forall\quad n \ge q.$$

Hence $\lim\limits_{n\to\infty}\left(\frac{b_1+b_2+\ldots+b_n}{n}\right) = 0$, which was to be proved.

**Remark.** *The converse of the above theorem may not be true.*

For example, $\langle a_n\rangle = \langle -1, 1, -1, 1, \ldots\rangle$ is not convergent

and $\quad a_1+a_2 = 0,\ a_1+a_2+a_3 = -1,\ a_1+a_2+a_3+a_4 = 0$, etc.

Now $\quad \frac{a_1+a_2+\ldots+a_n}{n} = 0$, if $n$ is even

$$= -\frac{1}{n}, \text{ if } n \text{ is odd.}$$

Hence $\lim\limits_{n\to\infty}\left(\frac{a_1+a_2+\ldots+a_n}{n}\right) = 0$, but $\lim\limits_{n\to\infty} a_n \ne 0$.

**Theorem 8. (Cauchy's second theorem on limits)**

*If* $\lim\limits_{n\to\infty} a_n = l$, *where* $a_n > 0\ \forall\ n \in \mathbf{N}$, *then* $\lim\limits_{n\to\infty}(a_1 a_2 \ldots a_n)^{\frac{1}{n}} = l$.

[D.U., Maths (H) 1998 ; Physics (H) 1994 ; B.A. (P) 1995]

**Proof.** $\lim\limits_{n\to\infty} a_n = l \Rightarrow \lim\limits_{n\to\infty}(\log a_n) = \log\left(\lim\limits_{n\to\infty} a_n\right) = \log l.$

By Cauchy's first theorem on limits, $\lim\limits_{n\to\infty}(\log a_n) = \log l$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{\log a_1+\log a_2+\ldots+\log a_n}{n}\right) = \log l$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{n}\log(a_1a_2\ldots.a_n)\right) = \log l$$

$$\Rightarrow \lim_{n\to\infty}\left[\log\left(a_1a_2\ldots a_n\right)^{\frac{1}{n}}\right] = \log l$$

$$\Rightarrow \log\left[\lim_{n\to\infty}\left(a_1a_2\ldots a_n\right)^{\frac{1}{n}}\right] = \log l.$$

Hence $\quad \lim\limits_{n\to\infty}(a_1a_2\ldots a_n)^{\frac{1}{n}} = l.$

**Corollary.** *Suppose $\langle a_n\rangle$ is a sequence of positive real numbers such*

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = a\ (a > 0). \text{ Then } \lim_{n\to\infty}(a_n)^{1/n} = a.$$

**Proof.** Let $a_0 = 1$. Then $\lim\limits_{n\to\infty}\frac{a_n}{a_{n-1}} = a.$

By Theorem 8, it follows that

$$\lim_{n\to\infty}\left[\frac{a_1}{a_0}\cdot\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdots\frac{a_n}{a_{n-1}}\right]^{\frac{1}{n}} = a.$$

Hence $\quad \lim\limits_{n\to\infty}(a_n)^{1/n} = a.$

## EXAMPLES

✓ **Example 37.** *Show that* $\lim\limits_{n\to\infty}\frac{1}{n}\left[1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right] = 0.$

**Solution.** Let $a_n = \frac{1}{n}$. Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{1}{n} = 0.$

Since $\quad \lim\limits_{n\to\infty} a_n = 0$, so by Cauchy's first theorem on limits,

$$\lim_{n\to\infty}\frac{1}{n}(a_1+a_2+\ldots+a_n) = 0.$$

Hence $\quad \lim\limits_{n\to\infty}\frac{1}{n}\left[1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right] = 0.$

✓ **Example 38.** *Show that*

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}}\right] = 1.$$

**Solution.** Let $\quad a_n = \frac{n}{\sqrt{n^2+n}} \Rightarrow \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}} = 1.$

By Cauchy's first theorem on limits, we have

$$\lim_{n\to\infty}\frac{a_1+a_2+\ldots+a_n}{n} = 1$$

$$= \lim_{n \to \infty} \frac{1}{n^2} + \lim_{n \to \infty} \left[ \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2} \right]$$

$= 0 + 0$, by Example 40

$= 0.$

**Example 42.** *If the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ converge to $A$ and $B$, respectively, then prove that*

$$\lim_{n \to \infty} \frac{1}{n} \left( a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1 \right) = AB.$$

[D.U., B.Sc. (G) 1989]

**Solution.** Let $a_n = A + c_n \ \forall \ n \in \mathbf{N}$. ...(1)

Then $a_n \to A \Rightarrow c_n \to 0.$

We know $c_n \to 0 \Leftrightarrow |c_n| \to 0.$ (See Example 16)

Let $p_n = |c_n| \ \forall \ n \in \mathbf{N}$. Then $\lim p_n = 0$.

By Cauchy's first theorem on limits, we have

$$\lim_{n \to \infty} \frac{1}{n} (p_1 + p_2 + \dots + p_n) = 0. \quad \text{...(2)}$$

Now $\frac{1}{n} (a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1)$

$= \frac{1}{n} [(A + c_1) b_n + (A + c_2) b_{n-1} + \dots + (A + c_n) b_1]$, using (1)

$$= \frac{A}{n} (b_1 + b_2 + \dots + b_n) + \frac{1}{n} (c_1 b_n + c_2 b_{n-1} + \dots + c_n b_1). \quad \text{...(3)}$$

Since $\lim_{n \to \infty} b_n = B$, by Cauchy's first theorem on limits,

$$\lim_{n \to \infty} \frac{1}{n} (b_1 + b_2 + \dots + b_n) = B. \quad \text{...(4)}$$

Again $\langle b_n \rangle$ is a convergent sequence $\Rightarrow \langle b_n \rangle$ is bounded and so there exists a positive number $k$ such that

$$|b_n| \le k \ \ \forall \ n. \quad \text{...(5)}$$

Using (5), we observe that

$$\left| \frac{1}{n} (c_1 b_n + c_2 b_{n-1} + \dots + c_n b_1) \right| \le \frac{k}{n} (|c_1| + |c_2| + \dots + |c_n|)$$

$$= k \left\{ \frac{1}{n} (p_1 + p_2 + \dots + p_n) \right\}, \quad (\because \ p_n = |c_n|)$$

$\to 0$ as $n \to \infty$, by (2).

$$\text{Thus} \ \lim_{n \to \infty} \left[ \frac{1}{n} (c_1 b_n + c_2 b_{n-1} + \dots + c_n b_1 \right] = 0. \quad \text{...(6)}$$

Using (4) and (6) in (3), we obtain

$$\lim_{n\to\infty} \frac{1}{n}(a_1b_n + a_2b_{n-1} + \dots + a_nb_1) = AB.$$

**Example 43.** *Prove that if a sequence $\{a_n\}$ of positive numbers converges to a limit $l$, then*

$$\lim_{n\to\infty} (a_1 a_2 \dots a_n)^{1/n} = l.$$

*Deduce that* $\lim_{n\to\infty} n^{1/n} = 1.$ [D.U., Maths (H) 1998]

**Solution.** See Theorem 8 for the first part.

(*ii*) We define a sequence $\langle a_n \rangle$ as follows :

$$a_1 = 1,\ a_n = \frac{n}{n-1} \quad \text{for} \quad n \ge 2.$$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n-1} = \lim_{n\to\infty} \frac{1}{1-(1/n)} = 1.$

By Cauchy's second theorem on limits,

$$\lim_{n\to\infty} (a_1 a_2 a_3 \dots a_n)^{1/n} = l$$

or $\lim_{n\to\infty} \left(1 \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n}{n-1}\right)^{1/n} = 1.$ Hence $\lim_{n\to\infty} (n)^{1/n} = 1.$

**Remark.** For more applications of Cauchy's second theorem on limits, we shall use an important limit

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e = 2.7182\dots$$

For the proof of this limit, see Example 88.

**Example 44.** *If* $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$, *then show that*

$$\lim_{n\to\infty} \left[\frac{2}{1}\left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \dots \left(\frac{n+1}{n}\right)^n\right]^{1/n} = e.$$

[D.U., Maths (H) 1993]

**Solution.** Let $a_n = \left(1 + \frac{1}{n}\right)^n.$ ...(1)

Then $a_n > 0\ \forall\, n \in \mathbf{N}$ and $\lim_{n\to\infty} a_n = e > 0.$

By Cauchy's second theorem on limits, we have

$$\lim_{n\to\infty} (a_1 a_2 a_3 \dots a_n)^{1/n} = e.$$

Hence $\lim_{n\to\infty} \left[\frac{2}{1}\left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \dots \left(\frac{n+1}{n}\right)^n\right]^{1/n} = e$, using (1).

**Example 45.** *Prove that* $\lim_{n\to\infty} \left(\frac{n^n}{n\,!}\right)^{1/n}$ *exists and equals e.*

[D.U., B.A. (P) 1999]

**Solution.** By Example 44, we have

$$e = \lim_{n\to\infty} \left[\frac{2}{1}\left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \dots \left(\frac{n+1}{n}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty} \left[\frac{(n+1)^n}{1.2.3 \dots n}\right]^{1/n} = \lim_{n\to\infty} \left[\frac{(n+1)^n}{n\,!}\right]^{1/n}$$

$$= \lim_{n\to\infty} (n+1)\left(\frac{1}{n\,!}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{n^n}{n\,!}\right)^{1/n}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \cdot \lim_{n\to\infty} \left(\frac{n^n}{n\,!}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{n^n}{n\,!}\right)^{1/n}.$$

Hence $\lim_{n\to\infty} \left(\frac{n^n}{n\,!}\right)^{1/n} = e.$

**Example 46.** *Prove that* $\lim_{n\to\infty} \left\{\frac{(3n)\,!}{(n\,!)^3}\right\}^{1/n} = 27.$

**Solution.** We have $a_n = \frac{(3n)\,!}{(n\,!)^3},\ a_{n+1} = \frac{(3n+3)\,!}{((n+1)\,!)^3}$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(3n+3)\,!}{((n+1)\,!)^3} \cdot \frac{(n\,!)^3}{(3n)\,!}$$

$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3}$$

$$= \lim_{n\to\infty} \frac{(3+3/n)(3+2/n)(3+1/n)}{(1+1/n)^3} = 27.$$

Using the Corollary of Theorem 8, we get

$$\lim_{n\to\infty} (a_n)^{1/n} = 27.$$

**Example 47.** *Prove that if*

$$a_n = \frac{1}{n}\{(n+1)(n+2)\dots(n+n)\}^{1/n},$$

*then* $\langle a_n \rangle$ *converges to* $4/e$. [D.U., B.A. (P) 1997]

**Theorem 10.** *If* $\langle a_n \rangle$ *be a sequence such that*

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l > 1, \text{ then } \lim_{n \to \infty} a_n = \infty.$$

**Proof.** Since $l > 1$, we can choose a positive number $\varepsilon$ such that

$$l - \varepsilon > 1 \quad \text{or} \quad k > 1, \text{ where } k = l - \varepsilon.$$

Since $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n} = l$, there exists a positive integer $m$ such that

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \varepsilon, \quad \forall \quad n \geq m$$

$$\Rightarrow \qquad l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon, \quad \forall \quad n \geq m.$$

Consider $\dfrac{a_{n+1}}{a_n} > l - \varepsilon = k \quad \forall \quad n \geq m.$

$$\therefore \qquad \frac{a_{n+1}}{a_n} > k \quad \forall \quad n \geq m.$$

Putting $n = m, m + 1, \ldots, (n - 1)$ successively and multiplying, we get

$$\left| \frac{a_n}{a_m} \right| > k^{n-m}, \text{ as done in Theorem 9}$$

$$\Rightarrow \qquad |a_n| > \frac{|a_m|}{k^m} k^n.$$

Now $k > 1 \Rightarrow k^n \to \infty$ as $n \to \infty$, by Example 24.

Hence $\lim\limits_{n \to \infty} a_n = \infty$.

## EXAMPLES

**Example 48.** (*i*) *Let* $\langle a_n \rangle$ *be a sequence of real numbers such that* $a_n \neq 0$ *for all n and* $\lim\limits_{n \to \infty} \left( \dfrac{a_{n+1}}{a_n} \right) = l$, *where* $|l| < 1$. *Show that* $\lim\limits_{n \to \infty} a_n = 0$.

(*ii*) *Deduce that* $\lim\limits_{n \to \infty} 2^{-n} n^2 = 0$. [D.U., Physics (H) 1995]

**Solution.** (*i*) See Theorem 9.

(*ii*) Let $a_n = \dfrac{n^2}{2^n}$ so that $a_{n+1} = \dfrac{(n+1)^2}{2^{n+1}}$.

$$\therefore \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \lim_{n \to \infty} \frac{1}{2} \left( \frac{n+1}{n} \right)^2$$

$$= \lim_{n \to \infty} \frac{1}{2} \left( 1 + \frac{1}{n} \right)^2 = \frac{1}{2} < 1.$$

Hence, by Theorem 9, $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} 2^{-n} n^2 = 0$.

**Example 49.** *Show that* $\lim\limits_{n \to \infty} (3^{-n} n^3) = 0.$ [D.U., B.C.A. 1999]

**Solution.** Let $a_n = \dfrac{n^3}{3^n}$ so that $a_{n+1} = \dfrac{(n+1)^3}{3^{n+1}}$.

$$\therefore \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \lim_{n \to \infty} \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 = \frac{1}{3} < 1.$$

Hence $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} 3^{-n} n^3 = 0.$ [Theorem 9]

**Example 50.** *Show that for any number x,* $\lim\limits_{n \to \infty} \dfrac{x^n}{n!} = 0.$

[D.U., Physics (H) 1999]

**Solution.** Let $a_n = \dfrac{x^n}{n!}$, so that $a_{n+1} = \dfrac{x^{n+1}}{(n+1)!}$.

$$\Rightarrow \quad \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}.$$

$$\therefore \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{x}{n+1} = 0 < 1.$$

Hence $\lim\limits_{n \to \infty} a_n = 0$ *i.e.*, $\lim\limits_{n \to \infty} \dfrac{x^n}{n!} = 0$ (Theorem 9).

**Example 51.** *Show that*

$$\lim_{n \to \infty} \frac{m(m-1)(m-2)\dots(m-n+1)}{n!} x^n = 0, \text{ if } |x| < 1.$$

**Solution.** Let $a_n = \dfrac{m(m-1)\dots(m-n+1)}{n!} x^n$, so that

$$a_{n+1} = \frac{m(m-1)\dots(m-n+1)(m-n)}{(n+1)!} x^{n+1}.$$

$$\text{Now} \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left\{ \frac{(m-n)}{(n+1)!} \cdot n! \cdot x \right\}$$

$$= \lim_{n \to \infty} \left( \frac{m-n}{n+1} \right) x = \lim_{n \to \infty} \frac{\left(\dfrac{m}{n}\right) - 1}{1 + \left(\dfrac{1}{n}\right)} \cdot x = -x.$$

$$\therefore \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = -x, \text{ where } |-x| = |x| < 1.$$

Hence $\lim\limits_{n \to \infty} a_n = 0$, by Theorem 9.

**ample 52.** *Prove that if $p > 0$, then* $\lim\limits_{n \to \infty} \dfrac{n^k}{(1+p)^n} = 0$,

*a fixed real number.* [D.U., Maths (H) 1992]

**Solution.** Let $a_n = \dfrac{n^k}{(1+p)^n}$, so that $a_{n+1} = \dfrac{(n+1)^k}{(1+p)^{n+1}}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^k \cdot \frac{1}{1+p} = \left(1 + \frac{1}{n}\right)^k \frac{1}{1+p}$$

$$\text{Now} \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{(1+p)} \cdot \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^k = \frac{1}{1+p}.$$

re $\quad \dfrac{1}{1+p} < 1 \quad (\because 1 + p > 1 \text{ as } p > 0)$

Hence $\lim\limits_{n \to \infty} a_n = 0$, by Theorem 9.

**Example 53.** *If $|x| > 1$ and $k > 0$, prove that* $\lim\limits_{n \to \infty} \dfrac{n^k}{x^n} = 0.$

**Solution.** Let $a_n = \dfrac{n^k}{x^n}$, so that $a_{n+1} = \dfrac{(n+1)^k}{x^{n+1}}$.

$$\therefore \quad \frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^k \frac{1}{x} = \left(1 + \frac{1}{n}\right)^k \cdot \frac{1}{x}$$

$$\text{Now} \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{x}, \text{ where } \left|\frac{1}{x}\right| = \frac{1}{|x|} < 1. \qquad (\because |x| > 1)$$

Hence $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \dfrac{n^k}{x^n} = 0$, by Theorem 9.

**Example 54.** *Show that* $\lim\limits_{n \to \infty} \dfrac{(1+y)^n}{n!} = 0$, *for all y.*

[D.U., Maths (H) 1992, 90]

**Solution.** Let $a_n = \dfrac{(1+y)^n}{n!}$, so that $a_{n+1} = \dfrac{(1+y)^{n+1}}{(n+1)!}$.

$$\text{Now} \quad \frac{a_{n+1}}{a_n} = (1+y) \frac{n!}{(n+1)!} = \frac{1+y}{n+1}.$$

$$\therefore \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1+y}{n+1} = 0 < 1.$$

Hence $\lim\limits_{n \to \infty} a_n = 0$ *i.e.*, $\lim\limits_{n \to \infty} \dfrac{(1+y)^n}{n!} = 0$, by Theorem 9.

### 3.8 Limit Points or Cluster Points

**Definition.** *A real number p is said to be a* **limit point** *or a* **cluster point** *of a sequence if every neighbourhood of p contains an infinite number of elements of the given sequence.*

*In other words, a real number p is a* **limit point** *of a sequence* $\langle a_n \rangle$ *if for any* $\varepsilon > 0$,

$$a_n \in \,]p-\varepsilon, p+\varepsilon[ \text{ for infinitely many values of } n.$$

**Remark.** *A real number p is* **not a limit point** *of a sequence* $\langle a_n \rangle$, *if there exists at least one neighbourhood of p which contains only* **finite** *number of elements of* $\langle a_n \rangle$.

**Illustrations**

1. The sequence $\langle 1+(-1)^n \rangle = \langle 0, 2, 0, 2, \dots \rangle$
has only two limit points *viz.* 0 and 2, since for any $\varepsilon > 0$,
$a_n = 0 \in \,]0-\varepsilon, 0+\varepsilon[$ for infinitely many values of $n$,
and $a_n = 2 \in \,]2-\varepsilon, 2+\varepsilon[$ for infinitely many values of $n$.
2. The sequence $\langle (-1)^n \rangle = \langle -1, 1, -1, 1, \dots \rangle$
has only two limit points $-1$ and 1.
3. The sequence $\langle \frac{1}{n} \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \rangle$
has only 0 as its limit point.
4. The sequence $\langle n \rangle = \langle 1, 2, 3, \dots \rangle$ has no limit point, since for any $n \in \mathbf{N}$, $]n-\frac{1}{10}, n+\frac{1}{10}[$ is a *nbd.* of $n$ which contains no element of the given sequence except $n$. Indeed if $p$ is any real number, then a *nbd.* $]p-\frac{1}{10}, p+\frac{1}{10}[$ of $p$ does not contain infinite number of elements of $\langle n \rangle$.
5. The sequence $\langle n^2 \rangle = \langle 1, 4, 9, 16, \dots \rangle$ has no limit point.
6. The sequence $\langle (-1)^n (1+\frac{1}{n}) \rangle = \langle -2, 1\frac{1}{2}, -1\frac{1}{3}, 1\frac{1}{4}, -1\frac{1}{5}, \dots \rangle$ has only two limit points 1 and $-1$.

**Theorem 11.** *If* $\lim\limits_{n \to \infty} a_n = l$, *then l is the unique limit point of* $\langle a_n \rangle$.

**Proof.** First of all we show that $l$ is a limit point of $\langle a_n \rangle$.

Since $\lim\limits_{n \to \infty} a_n = l$, therefore, for any $\varepsilon > 0$, there exists a positive integer $m$ such that

$$|a_n - l| < \varepsilon \;\forall\; n \geq m \text{ (which gives infinitely many values of } n).$$

*i.e.,* $\quad l-\varepsilon < a_n < l+\varepsilon$ for infinitely many values of $n$

*i.e.,* $\quad a_n \in \,]l-\varepsilon, l+\varepsilon[$ for infinitely many values of $n$.

Thus $l$ is a limit point of the sequence $\langle a_n \rangle$

Now we show that if $l'$ be any other limit point of $\langle a_n \rangle$, then $l = l'$.

Since $\lim\limits_{n \to \infty} a_n = l$, there exists a positive integer $i$ such that

$$|a_n - l| < \varepsilon/2 \;\;\forall\; n \geq i. \qquad \dots(1)$$

Since $l'$ is a limit point of $\langle a_n \rangle$, there exists a positive integer $j$ such that

$a_j \in \,]l' - \frac{1}{2}\varepsilon, l' + \frac{1}{2}\varepsilon[$ for $j > i$ (*i.e.*, for infinitely many values of $j$).

$$\Rightarrow \qquad |a_j - l'| < \varepsilon/2 \text{ for } j > i. \qquad \dots(2)$$

Since $j > i$, so by (1), $|a_j - l| < \varepsilon/2$. $\qquad \dots(3)$

Now $\quad |l - l'| = |(a_j - l') - (a_j - l)|$

$$\leq |a_j - l'| + |a_j - l| < \varepsilon/2 + \varepsilon/2 = \varepsilon, \text{ by (2) and (3).}$$

Since $\varepsilon$ is arbitrarily small and $|l - l'| < \varepsilon$, so $|l - l'| = 0$ *i.e.*, $l = l'$.

Hence $l$ is the unique limit point of $\langle a_n \rangle$.

**Remark.** *The converse of the above theorem may not be true i.e., a sequence having only one limit point may not converge.*

For example, the sequence $\langle 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \dots \rangle$ has exactly one limit point *viz.* 0, but the sequence is not convergent.

Indeed this sequence oscillates infinitely.

The following theorem guarantees the existence of limit points of any bounded sequence.

**Theorem 12. (Bolzano-Weierstrass Theorem)**

*Every bounded sequence has a limit point.* [D.U., Maths (H) 1999]

**Proof.** Let $\langle a_n \rangle$ be a bounded sequence.

Let $\quad S = \{a_n : n \in \mathbf{N}\}$ be its range.

Since the sequence is bounded, therefore, its range $S$ is also bounded.

**Case I.** Let $S$ be a finite set.

Then there must exist at least one element $\alpha \in S$ such that

$$a_n = \alpha \text{ for an infinite number of values of } n.$$

For any $\varepsilon > 0$, the *nbd.* $]\alpha-\varepsilon, \alpha+\varepsilon[$ of $\alpha$, contains $a_n = \alpha$, for an infinite number of values of $n$. Consequently, $\alpha$ is a limit point of $\langle a_n \rangle$.

**Case II.** Let $S$ be an infinite set.

The range $S$ being an infinite bounded set has a limit point, say $p$, (by Bolzano-Weierstrass Theorem). So each *nbd.* $]p-\varepsilon, p+\varepsilon[$ of $p$ contains an infinite number of elements of $S$

*i.e.,* $\quad a_n \in \,]p-\varepsilon, p+\varepsilon[$ for an infinite number of values of $n$.

Hence $p$ is a limit point of $\langle a_n \rangle$.

**Remark.** *An unbounded sequence may or may not have a limit point.*

For example, $\quad \langle n^2 \rangle = \langle 1, 4, 9, 16, \dots \rangle$
is an unbounded sequence having no limit point.

Again $\langle 1, 2, 1, 4, 1, 6, \dots \rangle$
is an unbounded sequence having 1 as its limit point.

**Ex. 1.** *Give an example of an unbounded sequence with exactly two limit points.* [D.U., Maths (H) 1999]

**Hint.** $\langle a_n \rangle = \langle 0, 1, 2, 0, 1, 2^2, 0, 1, 2^3, 0, 1, 2^4, \dots \rangle$ is an unbounded sequence having two limit points 0 and 1.

**Ex. 2.** *Give an example of a bounded sequence having two limit points.*

**Hint.** $\langle (-1)^n \rangle$ has two limits points $-1$ and 1.

**Ex. 3.** *Give an example of a bounded sequence having three limit points.*

**Hint.** $\langle 1, 3, 5, 1, 3, 5, 1, 3, 5, \dots \rangle$.

**Theorem 13.** *The set of limit points of a bounded sequence is bounded.* [D.U., Maths (H) 1999]

**Proof.** Let $\langle a_n \rangle$ be a bounded sequence. By Bolzano-Weierstrass Theorem, $\langle a_n \rangle$ must have a limit point. Let $S$ be the set of all limit points of $\langle a_n \rangle$. We shall show that $S$ is bounded. Since $\langle a_n \rangle$ is bounded, so there exist real numbers $m$ and $M$ such that

$$m \le a_n \le M \quad \forall \quad n \in \mathbf{N}. \qquad \dots(1)$$

It will be shown that $m \le x \le M \quad \forall \quad x \in S$.

Let, if possible, $p > M$ for some $p \in S$.

Now $p \in S \Rightarrow p$ is a limit point of $\langle a_n \rangle$.

As $p > M$, $p - M > 0$. Let $\varepsilon$ be a positive number such that

$$\varepsilon < p - M \quad \Rightarrow \quad M < p - \varepsilon. \qquad \dots(2)$$

From (1) and (2), $a_n < p - \varepsilon \ \forall \ n \in \mathbf{N}$.

Consequently, $\quad a_n \notin \, ]p - \varepsilon, p + \varepsilon[ \ \forall \ n \in \mathbf{N}$.

Thus $]p - \varepsilon, p + \varepsilon[$ is a *nbd.* of $p$ which does not contain any element of $\langle a_n \rangle$ and so $p$ is not a limit point of $\langle a_n \rangle$, which is a contradiction. Hence

$$x \le M \quad \forall \quad x \in S. \text{ Similarly, } m \le x \quad \forall \quad x \in S.$$

Hence $S$ is bounded.

**Theorem 14.** *Prove that a bounded sequence with a unique limit point is convergent.* [D.U., Maths (H) 1999]

**Proof.** Let $l$ be the unique limit point of a bounded sequence $\langle a_n \rangle$. Then for any $\varepsilon > 0$,

$$a_n \in \, ]l - \varepsilon, l + \varepsilon[ \text{ for infinitely many values of } n.$$

We claim that there exists only finitely many values of $n$, say $m_1, m_2, \dots, m_k$ such that $a_{m_1}, a_{m_2}, \dots, a_{m_k}$ do not belong to $]l - \varepsilon, l + \varepsilon[$ ; for otherwise, the infinitely many terms of the given sequence not belonging to $]l - \varepsilon, l + \varepsilon[$ will have a limit point other than $l$ (by Bolzano-Weierstrass Theorem), which is contrary to the given hypothesis.

Let $m - 1 = \max \{m_1, m_2, \dots, m_k\}$. It follows that

$$a_n \in \, ]l - \varepsilon, l + \varepsilon[ \quad \text{for all} \quad n \ge m$$

*i.e.* $\qquad |a_n - l| < \varepsilon \quad$ for all $\quad n \ge m$.

Hence $\langle a_n \rangle$ is convergent to $l$.

**emark.** *The conditions of Theorem 14 can not be relaxed.* [D.U., Maths (H) 1999]

r example, $\langle (-1)^n \rangle = \langle -1, 1, -1, 1, \dots \rangle$ is a bounded sequence two limit points $1, -1$ ; which is not convergent. Again , 4, 1, 6, ... ⟩ is an unbounded sequence having 1 as its only limit point is not convergent.

*heorems 11 and 14 can be restated as :*

**heorem 15.** *A necessary and sufficient condition for the convergence quence is that it is bounded and has a unique limit point.*

**emark.** *The above statement provides a second definition of a con- t sequence i.e.,*

*sequence is convergent if it is bounded and has a unique limit point.*

**Cauchy Sequence**

**efinition.** *A sequence $\langle a_n \rangle$ is said to be a* **Cauchy sequence** *if for $> 0$, there exists a positive integer $m$ such that*

$$|a_n - a_m| < \varepsilon, \text{ whenever } n \ge m.$$

**rations**

*The sequence $\langle 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots \rangle$ is a Cauchy sequence.*

Let $\varepsilon > 0$ be given.

If $n > m$, then $\left| \dfrac{1}{n} - \dfrac{1}{m} \right| = \dfrac{n - m}{nm} < \dfrac{1}{m} < \varepsilon$, if $m > \dfrac{1}{\varepsilon}$.

Let $m$ be a positive integer $> 1/\varepsilon$. Then

$$\left| \frac{1}{n} - \frac{1}{m} \right| < \varepsilon, \text{ for } n \ge m.$$

Hence $\langle \frac{1}{n} \rangle$ is a Cauchy sequence.

*The sequence $\langle a_n \rangle = \langle \dfrac{n}{n+1} \rangle$ is a Cauchy sequence.*

Let $\varepsilon > 0$ be given. If $n > m$, then

$$|a_n - a_m| = \left| \frac{n}{n+1} - \frac{m}{m+1} \right| = \frac{n - m}{(n+1)(m+1)} < \frac{1}{m}$$

or $\quad |a_n - a_m| < \dfrac{1}{m} < \varepsilon$, if $m > \dfrac{1}{\varepsilon}$.

Let $m$ be a positive integer $> 1/\varepsilon$. Then $|a_n - a_m| < \varepsilon$, for $n \ge m$.

Hence $\langle \dfrac{n}{n+1} \rangle$ is a Cauchy sequence.

3. *The sequence $\langle n^2 \rangle$ is not a Cauchy sequence.*

If $n > m$, then $n^2 - m^2 = (n - m)(n + m) > 2m > 1$, whatever $m$ may be. Taking $\varepsilon = \frac{1}{2}$, we see that there is no positive integer $m$ such that

$$|n^2 - m^2| < \varepsilon \text{ for all } n \ge m.$$

**4.** *The sequence* $\langle a_n \rangle = \langle (-1)^n \rangle = \langle -1, 1, -1, 1, \ldots \rangle$ *is not a Cauchy sequence*, for if $\langle a_n \rangle$ is a Cauchy sequence, then for $\varepsilon = 1$, there exists a positive integer $m$ such that

$$| a_n - a_m | < 1 \;\forall\; n \geq m.$$

If $m$ is an even integer, then $a_m = 1$.

We can choose $n = 2m + 1 > m$ for which $a_n = -1$ and

$$| a_n - a_m | = | -1 - 1 | = 2 \nless 1, \text{ a contradiction.}$$

If $m$ is an odd integer, then $a_m = -1$.

We can choose $n = 2m > m$ for which $a_n = 1$ and

$$| a_n - a_m | = | 1 + 1 | = 2 \nless 1, \text{ a contradiction.}$$

Hence $\langle (-1)^n \rangle$ is not a Cauchy sequence.

**Theorem 16.** *Prove that every Cauchy sequence is bounded. The converse need not be true.* [D.U., B.Sc. (G) 1991]

**Proof.** Let $\langle a_n \rangle$ be a Cauchy sequence.

For $\varepsilon = 1$, there exists a positive integer $m$ such that

$$| a_n - a_m | < 1, \quad \text{whenever } n \geq m,$$

or

$$a_m - 1 < a_n < a_m + 1, \quad \text{whenever } n \geq m.$$

Let $k = \min \{a_1, a_2, \ldots, a_{m-1}, a_m - 1\}$,

$K = \max \{a_1, a_2, \ldots, a_{m-1}, a_m + 1\}$.

Then $k \leq a_n \leq K$ for all $n$.

Hence $\langle a_n \rangle$ is bounded.

**Remark.** *The converse of the above theorem is not true.*

For example, the sequence $\langle a_n \rangle = \langle (-1)^n \rangle$ is bounded, but is not a Cauchy sequence.

**Theorem 17. (Cauchy's General Principle of Convergence)**

*A sequence of real numbers converges if and only if it is a Cauchy ence.* [D.U., B.Sc. (G) 1993, 92 ; Maths (H) 1995 ; Physics (H) 1993 ; B.A. (P) 1996, 91]

**Proof.** *Condition is necessary.*

Let $\langle a_n \rangle$ be a convergent sequence and $\lim\limits_{n \to \infty} a_n = l$. We shall show that $\rangle$ is a Cauchy sequence. Since $\lim a_n = l$, so for $\varepsilon > 0$, there exists a positive ger $m$ such that

$$| a_n - l | < \varepsilon/2, \quad \text{whenever } n \geq m.$$

For $n = m$, we have $| a_m - l | < \varepsilon/2$.

Now $| a_n - a_m | = | (a_n - l) - (a_m - l) | \leq | a_n - l | + | a_m - l |$,

$< \varepsilon/2 + \varepsilon/2$, whenever $n \geq m$.

$\therefore \quad | a_n - a_m | < \varepsilon \;\forall\; n \geq m.$

Hence $\langle a_n \rangle$ is a Cauchy sequence.

*Condition is sufficient*

Suppose $\langle a_n \rangle$ is Cauchy sequence. We shall show that $\langle a_n \rangle$ is a con- nt sequence. Since every Cauchy sequence is bounded, therefore, is bounded. Also, since every bounded sequence has a limit point, fore, $\langle a_n \rangle$ has a limit point, say $l$. We proceed to show that $a_n \to l$. Let be given.

Since $\langle a_n \rangle$ is a Cauchy sequence, therefore, there exists a positive r $m$ such that

$$| a_n - a_m | < \varepsilon/3, \quad \text{whenever } n \geq m. \qquad \ldots(1)$$

Also, since $l$ is a limit point of $\langle a_n \rangle$, therefore, there exists a positive r $k > m$, such that

$$| a_k - l | < \varepsilon/3. \qquad \ldots(2)$$

Since $k > m$, therefore, by (1)

$$| a_k - a_m | < \varepsilon/3. \qquad \ldots(3)$$

Now $| a_n - l | = | (a_n - a_m) + (a_m - a_k) + (a_k - l) |$

$\leq | a_n - a_m | + | a_m - a_k | + | a_k - l |$

$< \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} \;\forall\; n \geq m$, using (1), (2), (3)

$\therefore \quad | a_n - l | < \varepsilon \quad \forall \quad n \geq m.$

Hence the sequence $\langle a_n \rangle$ converges to $l$.

**Remark.** The above theorem may be restated as :

*A necessary and sufficient condition for the convergence of a sequence* $\rangle$ *is that, for each* $\varepsilon > 0$, *there exists a positive integer m such that*

$$| a_{n+p} - a_n | < \varepsilon \quad \forall \quad n > m \text{ and } p \geq 1. \qquad \ldots(1)$$

**Proof.** If $\langle a_n \rangle$ converges, then for $\varepsilon > 0$, there exists a positive integer such that (Theorem 17)

$$| a_n - a_m | < \varepsilon \quad \forall \quad n \geq m.$$

Now $| a_{n+p} - a_n | = | (a_{n+p} - a_m) + (a_m - a_n) |$, when $p \geq 1$

$\leq | a_{n+p} - a_m | + | a_m - a_n |$

$< \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} \quad \forall \quad n \geq m \qquad (\because\; n + p > m)$

$\therefore \quad | a_{n+p} - a_n | < \varepsilon \quad \forall \quad n \geq m$ and $p \geq 1$.

Conversely, let (1) be given. Taking $n = m$, we obtain

$| a_{m+p} - a_m | < \varepsilon$, where $p \geq 1$

$\Rightarrow \quad | a_k - a_m | < \varepsilon \quad \forall \quad k > m$

$\Rightarrow \langle a_n \rangle$ is a Cauchy sequence and so $\langle a_n \rangle$ converges.

## EXAMPLES

**Example 55.** *Show that the sequence* $\langle a_n \rangle$ *defined by* $a_n = (-1)^n/n$ *converges.*

**Solution.** If we show that $\langle a_n \rangle$ is a Cauchy sequence, then by Cauchy's General Principle of Convergence, $\langle a_n \rangle$ converges.

Let $\varepsilon > 0$. If $n \geq m$, then $1/n \leq 1/m$ and

$$|a_n - a_m| = \left| \frac{(-1)^n}{n} - \frac{(-1)^m}{m} \right| \leq \left| \frac{(-1)^n}{n} \right| + \left| \frac{(-1)^m}{m} \right|$$

$$\leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{m} + \frac{1}{m} = \frac{2}{m} < \varepsilon, \text{ if } m > \frac{2}{\varepsilon}.$$

Let $m$ be a positive integer $> 2/\varepsilon$. Then

$$|a_n - a_m| \quad \forall \quad n \geq m.$$

So $\langle a_n \rangle$ is a Cauchy sequence and hence convergent.

**Example 56.** *Use Cauchy's General Principle of Convergence to show that the sequence* $\langle n/n+1 \rangle$ *converges.*

**Solution.** By Illustration 2 of Section 3.9 ;

$\langle \frac{n}{n+1} \rangle$ is a Cauchy sequence and hence convergent.

**Example 57.** *Show that the sequence* $\langle a_n \rangle$, *where*

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \text{ is not convergent.}$$

[D.U., B.Sc. (G) 1994 ; B.A. (P) 1999, 92]

**Solution.** Suppose that the given sequence is convergent.

By Cauchy's General Principle of Convergence, for $\varepsilon = \frac{1}{2}$, there exists a positive integer $m$ such that

$$|a_n - a_m| < \tfrac{1}{2} \; \forall \; n \geq m.$$

In particular for $n = 2m$, $|a_{2m} - a_m| < \frac{1}{2}$. ...(1)

Now

$$|a_{2m} - a_m| = \left| \left( 1 + \frac{1}{2} + \ldots + \frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{2m} \right) - \left( 1 + \frac{1}{2} + \ldots + \frac{1}{m} \right) \right|$$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m}$$

$$> \frac{1}{2m} + \frac{1}{2m} + \ldots + \frac{1}{2m} \; (m \text{ terms}) = \frac{1}{2m} \cdot m = \frac{1}{2}.$$

$$\left( \because \; m + 1 < m + m \Rightarrow \frac{1}{m+1} > \frac{1}{2m} \text{ etc.} \right)$$

$$\therefore \quad |a_{2m} - a_m| > \tfrac{1}{2}.$$

This contradicts (1). Hence the given sequence is not convergent.

**Example 58.** *Show that the sequence* $\langle a_n \rangle$ *defined by*

$$a_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$$

*oes not converge.* [D.U., Maths (H) 1998, 1992 ; B.A. (P) 1999, 1994 ; Physics (H) 1995 ; B.Sc. (G) 1992, 90]

**Solution.** Let, if possible, $\langle a_n \rangle$ be convergent.

By Cauchy's General Principle of Convergence, for $\varepsilon = \frac{1}{4}$, there exists positive integer $m$ such that $|a_n - a_m| < \frac{1}{4} \; \forall \; n \geq m$.

In particular, for $n = 2m$, $|a_{2m} - a_m| < \frac{1}{4}$. ...(1)

Now $|a_{2m} - a_m|$

$$= \left| \left( 1 + \frac{1}{3} + \ldots + \frac{1}{2m-1} + \frac{1}{2m+3} + \ldots + \frac{1}{2m+2m-1} \right) - \left( 1 + \frac{1}{3} + \ldots + \frac{1}{2m-1} \right) \right|$$

$$= \frac{1}{2m+1} + \frac{1}{2m+3} + \ldots + \frac{1}{2m+2m-1} \; (m \text{ terms})$$

$$> \frac{1}{2m+2m} + \frac{1}{2m+2m} + \ldots + \frac{1}{2m+2m} \; (m \text{ terms}) = \frac{m}{4m} = \frac{1}{4}.$$

$$(\because \; 2m + 1 < 2m + 2m \text{ etc.})$$

Thus $|a_{2m} - a_m| > \frac{1}{4}$, which contradicts (1).

Hence the given sequence $\langle a_n \rangle$ is not convergent.

**Example 59.** *Apply Cauchy's principle of convergence to prove that the sequence* $\langle a_n \rangle$ *defined by*

$$a_n = 1 + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3n-2} \text{ is not convergent.}$$

[D.U., Maths (H) 1996 ; B.Sc. (G) 1997, 95]

**Solution.** Let, if possible, $\langle a_n \rangle$ be convergent. By Cauchy's principle of convergence, for $\varepsilon = \frac{1}{6}$, there exists a positive integer $m$ such that

$$|a_n - a_m| < \frac{1}{6} \text{ for } n \geq m.$$

In particular, for $n = 2m$, $|a_{2m} - a_m| < \frac{1}{6}$. ...(1)

Now $|a_{2m} - a_m|$

$$= \left| \left( 1 + \frac{1}{4} + \ldots + \frac{1}{3m-2} + \frac{1}{3m+1} + \frac{1}{3m+4} + \ldots + \frac{1}{6m-2} \right) - \left( 1 + \frac{1}{4} + \ldots + \frac{1}{3m-2} \right) \right|$$

$$= \frac{1}{3m+1} + \frac{1}{3m+4} + \dots + \frac{1}{6m-2}$$

$$> \frac{1}{6m} + \frac{1}{6m} + \dots + \frac{1}{6m} = \frac{m}{6m} = \frac{1}{6}.$$

$\therefore \quad |a_{2m} - a_m| > \frac{1}{6}.$ ...(2)

(1) and (2) contradict each other. Hence $\langle a_n \rangle$ is not convergent.

**Example 60.** *Show that the sequence*

$$\langle 1 + \tfrac{1}{4} + \tfrac{1}{7} + \dots + \tfrac{1}{3n-2} \rangle$$

*is not convergent, while*

$$\langle \tfrac{1}{n} ( 1 + \tfrac{1}{4} + \tfrac{1}{7} + \dots + \tfrac{1}{3n-2}) \rangle$$

*is convergent.* [D.U., Maths (H) 1996]

**Solution.** (*i*) See Example 59.

(*ii*) Let $a_n = \frac{1}{3n-2}$. Then $\lim\limits_{n \to \infty} a_n = 0$.

By Cauchy's first theorem on limits, we have

$$\lim_{n \to \infty} \frac{a_1 + a_2 + a_3 + \dots + a_n}{n} = 0.$$

Hence $\lim\limits_{n \to \infty} \left[ \frac{1}{n} \left( 1 + \frac{1}{4} + \frac{1}{7} + \dots + \frac{1}{3n-2} \right) \right] = 0.$

**Example 61.** *Show that the sequence*

$\langle 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \rangle$ *is not convergent, while*

$\langle \frac{1}{n} (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}) \rangle$ *is convergent.* by cauchy's 1st meth:

(*b*) *Show that the sequence*

$\langle 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1} \rangle$ *is not convergent, while*

$\langle \frac{1}{n} (1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1}) \rangle$ *is convergent.*

**Please try yourself.**

**Example 62.** *Show that the sequence* $\langle a_n \rangle$ *defined as*

$$a_n = 1 + \frac{1}{5} + \frac{1}{9} + \dots + \frac{1}{4n-3}$$

*does not converge, while* $\langle b_n \rangle$ *defined as* $b_n = \frac{1}{n} a_n$ *converges to* 0.

[D.U., B.C.A. 1999]

**Solution.** Let, if possible, $\langle a_n \rangle$ be convergent. By Cauchy's principle of convergence, for $\varepsilon = \frac{1}{8}$, there exists a positive integer $m$ such that

$$|a_n - a_m| < \tfrac{1}{8} \ \forall \ n \geq m.$$

In particular, $|a_{2m} - a_m| < \frac{1}{8}$. (Take $n = 2m > m$) ...(1)

Now $|a_{2m} - a_m|$

$$= \left| \left( 1 + \frac{1}{5} + \dots + \frac{1}{4m-3} + \frac{1}{4m+1} + \frac{1}{4m+5} + \dots + \frac{1}{8m-3} \right) - \left( 1 + \frac{1}{5} + \dots + \frac{1}{4m-3} \right) \right|$$

$$= \frac{1}{4m+1} + \frac{1}{4m+5} + \dots + \frac{1}{8m-3}$$

$$> \frac{1}{8m} + \frac{1}{8m} + \dots + \frac{1}{8m} = \frac{m}{8m} = \frac{1}{8}.$$

$\therefore \quad |a_{2m} - a_m| > \frac{1}{8}.$ ...(2)

(1) and (2) contradict each other. Hence $\langle a_n \rangle$ is not convergent.

(*ii*) We have $b_n = \frac{1}{n} a_n = \frac{1}{n} \left( 1 + \frac{1}{5} + \frac{1}{9} + \dots + \frac{1}{4n-3} \right)$

Let $t_n = \frac{1}{4n-3}$, so that $\lim\limits_{n \to \infty} t_n = 0$.

By Cauchy's first theorem on limits,

$$\lim_{n \to \infty} \frac{1}{n} \left( t_1 + t_2 + \dots + t_n \right) = 0$$

$$\lim_{n \to \infty} \frac{1}{n} \left( 1 + \frac{1}{5} + \frac{1}{9} + \dots + \frac{1}{4n-3} \right) = 0.$$

Hence $\lim\limits_{n \to \infty} b_n = 0.$

**Example 63.** *Show that the sequence* $\langle a_n \rangle$ *defined as*

$$a_n = 1 + \frac{1}{6} + \frac{1}{11} + \dots + \frac{1}{5n-4}$$

[D.U., B.C.A. 1998]

*not Cauchy.*

**Solution.** If $\langle a_n \rangle$ is a Cauchy sequence, then for $\varepsilon = \frac{1}{10}$, there exists positive integer $m$ such that $|a_n - a_m| < \frac{1}{10} \ \forall \ n \geq m.$

In particular, for $n = 2m$, $|a_{2m} - a_m| < \frac{1}{10}$. ...(1)

Now $|a_{2m} - a_m|$

$$= \left| \left( 1 + \frac{1}{6} + \dots + \frac{1}{5m-4} + \frac{1}{5m+1} + \frac{1}{5m+3} + \dots + \frac{1}{10m-4} \right) - \left( 1 + \frac{1}{6} + \dots + \frac{1}{5m-4} \right) \right|$$

$$= \frac{1}{5m+1} \cdot \frac{1}{5m+3} + \dots + \frac{1}{10m-4}$$

$$> \frac{1}{10m} + \frac{1}{10m} + \dots + \frac{1}{10m} = \frac{m}{10m} = \frac{1}{10}.$$

...(2)

$\therefore \qquad |a_{2m} - a_m| > \frac{1}{10}.$

Since (1) and (2) contradict each other, $\langle a_n \rangle$ is not a Cauchy sequence.

**Remark.** Since $\langle a_n \rangle$ is not a Cauchy sequence, $\langle a_n \rangle$ is not convergent. Hence the sequence $\langle a_n \rangle$ of Example 63 is not convergent.

**Example 64.** *Show that the sequence $\langle a_n \rangle$ defined as*

$$a_n = 1 + \frac{1}{7} + \frac{1}{13} + \dots + \frac{1}{6n-5}$$

*is not convergent.*

**Hint.** Take $\varepsilon = \frac{1}{12}$.

**Example 65.** *Show that the sequence $\langle a_n \rangle$ defined as*

$$a_n = \frac{1}{9} + \frac{1}{13} + \frac{1}{17} + \dots + \frac{1}{4n+5}$$

[B.C.A. Sample Problem]

*is not a Cauchy sequence.*

**Please try yourself.**

**Example 66.** *Use Cauchy's criterion of convergence of sequences to discuss the convergence of the sequence $\langle a_n \rangle$,*

$$a_n = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \dots + \frac{1}{n\,!}.$$

**Solution.** By Cauchy's criterion of convergence, a real sequence converges iff it is a Cauchy sequence. We shall show that the given sequence $\langle a_n \rangle$ is a Cauchy sequence, so that $\langle a_n \rangle$ is convergent. For $n \geq m$, we have

$$|a_n - c_m| = \left| \left( 1 + \frac{1}{1\,!} + \dots + \frac{1}{m\,!} + \frac{1}{(m+1)\,!} + \frac{1}{(m+2)\,!} + \dots + \frac{1}{n\,!} \right) - \left( 1 + \frac{1}{1\,!} + \dots + \frac{1}{m\,!} \right) \right|$$

$$= \frac{1}{(m+1)\,!} + \frac{1}{(m+2)\,!} + \dots + \frac{1}{n\,!}$$

$$\leq \frac{1}{2^m} + \frac{1}{2^{m+1}} + \dots + \frac{1}{2^{n-1}} \qquad (\because n\,! \geq 2^{n-1} \ \forall\ n)$$

$$= \frac{1}{2^m} \left[ 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{(n-m)-1}} \right]$$

$$= \frac{1}{2^m} \frac{[1 - (\frac{1}{2})^{n-m}]}{1 - (\frac{1}{2})} = \frac{1}{2^{m-1}} - \frac{1}{2^{n-1}} \leq \frac{1}{2^{m-1}}$$

$$\left( \because n \geq m \Rightarrow n - 1 \geq m - 1 \Rightarrow 2^{n-1} \geq 2^{m-1} \Rightarrow \frac{1}{2^{n-1}} \leq \frac{1}{2^{m-1}} \right)$$

For any $\varepsilon > 0$, $|a_n - a_m| < \varepsilon$, if $\frac{1}{2^{m-1}} < \varepsilon$ or $2^{m-1} > \frac{1}{\varepsilon}$

$(m-1) \log 2 > \log (1/\varepsilon)$ or $m > \{\log (1/\varepsilon) (\log 2)^{-1} + 1\} = N$, (say)

$\therefore \quad |a_n - a_m| < \varepsilon \quad \forall \quad n \geq m$, where $m$ is a positive integer $\geq N$.

Hence $\langle a_n \rangle$ is a Cauchy sequence and so $\langle a_n \rangle$ is convergent.

**Example 67.** *Let $\langle a_n \rangle$ be a sequence of real numbers and for each* ε **N**, *let*

$$s_n = a_1 + a_2 + \dots + a_n\,,\ t_n = |a_1| + |a_2| + \dots + |a_n|. \qquad \text{...(1)}$$

*Prove that if $\langle t_n \rangle$ is a Cauchy sequence, then so is $\langle s_n \rangle$.*

**Solution.** Since $\langle t_n \rangle$ is a Cauchy sequence, for any $\varepsilon > 0$, there exists positive integer $m$ such that $|t_n - t_m| < \varepsilon\ \forall\ n \geq m$

or $\quad |\{|a_1| + \dots + |a_m| + |a_{m+1}| + \dots + |a_n|\} - \{|a_1| + |a_2| + \dots + |a_m|\}| < \varepsilon \ \forall \ n \geq m$

$\quad ||a_{m+1}| + |a_{m+2}| + \dots + |a_n|| < \varepsilon \quad \forall \quad n \geq m$

$\quad |a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \varepsilon \quad \forall \quad n \geq m. \qquad$ ...(2)

Now $\quad |s_n - s_m| = |a_{m+1} + a_{m+2} + \dots + a_n|$, by (1)

$\leq |a_{m+1}| + |a_{m+2}| + \dots + |a_n|$

$< \varepsilon \quad \forall \quad n \geq m$, by (2).

$\therefore \qquad |s_n - s_m| < \varepsilon \quad \forall \quad n \geq m.$

Hence $\langle s_n \rangle$ is a Cauchy sequence.

**Example 68.** *Show that the sum and the product of two Cauchy quences are Cauchy sequences.*

**Solution.** Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be Cauchy sequences. We shall show hat $\langle a_n + b_n \rangle$ and $\langle a_n b_n \rangle$ are also Cauchy sequences.

(*i*) Since $\langle a_n \rangle$ and $\langle b_n \rangle$ are Cauchy sequences, therefore, for $\varepsilon > 0$, here exist positive integers $m_1$ and $m_2$ such that

$$|a_n - a_{m_1}| < \varepsilon/2 \quad \forall \quad n \geq m_1,$$

$$|b_n - b_{m_2}| < \varepsilon/2 \quad \forall \quad n \geq m_2.$$

If $m = \max (m_1, m_2)$, then

$|a_n - a_m| < \varepsilon/2$ and $|b_n - b_m| < \varepsilon/2 \quad \forall \quad n \geq m. \qquad$ ...(1)

Now $|(a_n + b_n) - (a_m + b_m)| = |(a_n - a_m) + (b_n - b_m)|$

$\leq |a_n - a_m| + |b_n - b_m|$

$< \varepsilon/2 + \varepsilon/2 \ \forall\ n \geq m$, by (1).

$\therefore \quad |(a_n + b_n) - (a_m + b_m)| < \varepsilon \ \forall \ n \geq m.$

Hence $\langle a_n + b_n \rangle$ is a Cauchy sequence.

(*ii*) Since every Cauchy sequence is bounded, so there exists a positive number $k$ such that

$$|a_n| \le k \quad \forall \; n. \qquad \text{...(2)}$$

Since $\langle a_n \rangle$ and $\langle b_n \rangle$ are Cauchy sequences, we can find a suitable positive integer $m$ such that

$$|b_n - b_m| < \frac{\varepsilon}{2k} \;\; \forall \; n \ge m, \qquad \text{...(3)}$$

and
$$|a_n - a_m| < \frac{\varepsilon}{2(|b_m|+1)} \quad \forall \; n \ge m. \qquad \text{...(4)}$$

Now
$$\begin{aligned}|a_nb_n - a_mb_m| &= |a_n(b_n - b_m) + b_m(a_n - a_m)| \\ &\le |a_n||b_n - b_m| + |b_m||a_n - a_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\left(\frac{|b_m|}{|b_m|+1}\right) \quad \forall \; n \ge m \text{ ; by (2), (3), (4)}\end{aligned}$$

$$\therefore \qquad |a_nb_n - a_mb_m| < \varepsilon \; \forall \; n \ge m. \qquad \left(\because \frac{|b_m|}{|b_m|+1} < 1\right)$$

Hence $\langle a_nb_n \rangle$ is a Cauchy sequence.

## EXERCISES

1. Show that $\langle \frac{1}{n\,!} \rangle$ is a Cauchy sequence.

   **[Hint.** For $n \ge m$, $|a_n - a_m| = \left|\frac{1}{n\,!} - \frac{1}{m\,!}\right| \le \frac{1}{n\,!} + \frac{1}{m\,!} \le \frac{1}{2^{n-1}} + \frac{1}{2^{m-1}}$

   $$\le \frac{1}{2^{m-1}} + \frac{1}{2^{m-1}} = \frac{2}{2^{m-1}} = \frac{1}{2^{m-2}} < \varepsilon.$$

   Now proceed like Example 66.]
2. Show that $\langle (-1)^n n \rangle$ is not a Cauchy sequence.
3. Show that $\langle 2^n \rangle$ is not a Cauchy sequence.

   **[Hint.** Let, if possible, $\langle 2^n \rangle$ be a Cauchy sequence. Then for $\varepsilon = \frac{1}{2}$, there exists a positive integer $m$ such that $|2^n - 2^m| < \frac{1}{2} \; \forall \; n \ge m$. In particular, $|2^{2m} - 2^m| < \frac{1}{2}$ or $2^m(2^m - 1) < \frac{1}{2}$. But $2^m(2^m - 1) \ge 2$ for all positive integers $m$.]
4. Show that the sequence $\langle n^2 \rangle$ is not a Cauchy sequence.

   **[Hint.** See Illustration 3 of Section 3.9] [D.U., Maths (H) 1994]
5. Show that the sequence $\langle a_n \rangle$, $a_n = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \dots + \frac{1}{n\,!}$

   is a Cauchy sequence.

   **[Hint.** See Example 66]

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6. Prove that every convergent sequence of real numbers is a Cauchy sequence. Is its converse true ? [D.U., Maths (H) 1999]

   **[Hint.** Prove the necessary condition of Theorem 17. Its converse is also true.]
7. Define the terms 'Cauchy sequence' and 'convergent sequence'. Prove that every convergent sequence is a Cauchy sequence.

   [D.U., Physics (H) 1995]
8. Prove that every Cauchy sequence of real numbers is convergent.

   **[Hint.** Prove the sufficient condition of Theorem 17.]
9. Define a Cauchy sequence and prove that every Cauchy sequence converges. Hence, show that $\langle a_n \rangle$, where $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ is not a Cauchy sequence. [D.U., B.Sc. (G) 1994]
10. Define a Cauchy sequence of real numbers. Give an example of a bounded sequence which is not a Cauchy sequence.

    **[Hint.** $\langle (-1)^n \rangle$ is a bounded sequence, which is not a Cauchy sequence.]
11. If $\langle a_n \rangle$ and $\langle b_n \rangle$ are Cauchy sequences, show that $\langle a_n - b_n \rangle$ is a Cauchy sequence.

    **[Hint.** Proceed like Example 68.]
12. If $\langle a_n \rangle$ is a Cauchy sequence in **R** such that $a_n \ne 0$ for each $n$, show that $\langle 1/a_n \rangle$ is also a Cauchy sequence.

    **[Hint.** For $n \ge m$, $\left|\frac{1}{a_n} - \frac{1}{a_n}\right| = \frac{|a_m - a_n|}{|a_n||a_m|}$.

    Since $\langle a_n \rangle$ is a Cauchy sequence, $\langle a_n \rangle$ is convergent and so there exists some $k > 0$ such that $|a_n| > k$ for $n \ge m$. Also $|a_m| > k$. Since $\langle a_n \rangle$ is a Cauchy sequence, $|a_n - a_m| < k^2 \varepsilon \;\; \forall \; n \ge m$.]
13. Show that the sequence $\langle a_n \rangle$, where

    $$a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n-1}}{n}$$

    is a Cauchy sequence.

### 3.10 Monotonic Sequences

**Definition 1.** *A sequence* $\langle a_n \rangle = \langle a_1, a_2, a_3, \dots, a_n, \dots \rangle$ *is said to be* **monotonically increasing,** *if* $a_{n+1} \ge a_n \;\; \forall \; n \in \mathbf{N}$

*i.e.*, $\qquad a_2 \ge a_1 \,,\; a_3 \ge a_2 \,,\; a_4 \ge a_3$ *and so on.*

*Equivalently, a sequence* $\langle a_n \rangle$ *is said to be* **monotonically increasing,** *if* $a_n \ge a_m$, *whenever* $n \ge m$.

**Definition 2.** *A sequence* $\langle a_n \rangle = \langle a_1, a_2, a_3, \dots, a_n, \dots \rangle$ *is said to be* **monotonically decreasing,** *if* $a_{n+1} \le a_n \;\; \forall \; n \in \mathbf{N}$

*i.e.*, $\qquad a_2 \le a_1,\; a_3 \le a_2,\; a_4 \le a_3$ *and so on .*

*Equivalently, a sequence* $\langle a_n \rangle$ *is said to be* **monotonically decreasing**, *if* $a_n \le a_m$, *whenever* $n \ge m$.

**Definition 3.** *A sequence* $\langle a_n \rangle$ *is said to be* **monotonic** *if it is either monotonically increasing or monotonically decreasing.*

**Illustrations**

1. The sequence $\langle n^2 \rangle = \langle 1, 4, 9, 6, \ldots \rangle$
   is monotonically increasing.
2. The sequence $\langle 2^n \rangle = \langle 2, 4, 8, 16, \ldots \rangle$
   is monotonically increasing.
3. The sequence $\langle \frac{1}{n} \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$
   is monotonically decreasing.
4. The sequence $\langle -n \rangle = \langle -1, -2, -3, \ldots \rangle$
   is monotonically decreasing.
5. The sequence $\langle 1, -1, 1, -1, 1, -1, \ldots \rangle$
   is neither monotonically increasing nor monotonically decreasing.

$\therefore \qquad \langle (-1)^{n+1} \rangle$ is not a monotonic sequence.

**Theorem 18.** *A necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.*

[D.U., B.Sc. (G) 1992 ; Maths (H) 1992]

**Proof.** *The condition is necessary.*

Let $\langle a_n \rangle$ be a monotonic and convergent sequence.

We know that every convergent sequence is bounded.

Hence $\langle a_n \rangle$ is bounded.

*The condition is sufficient.*

Let $\langle a_n \rangle$ be a monotonic and bounded sequence.

Since $\langle a_n \rangle$ is monotonic, we may suppose that $\langle a_n \rangle$ is monotonically increasing. We shall prove that $\langle a_n \rangle$ is convergent.

Let $S = \{a_n : n \in \mathbf{N}\}$ be the range of $\langle a_n \rangle$.

Then $S$ is bounded above, since the sequence $\langle a_n \rangle$ is bounded above. By order completeness property, $S$ has the least upper bound.

Let $p = l.u.b.\ S$. We shall show that $\langle a_n \rangle$ converges to $p$. Let $\varepsilon > 0$ be any number. Since $p - \varepsilon < p$, $p - \varepsilon$ cannot be an upper bound of $S$ and so there exists some $a_m \in S$ such that $a_m > p - \varepsilon$.

Since $\langle a_n \rangle$ is monotonically increasing, therefore

$$a_n \ge a_m \quad \forall\, n \ge m$$

$$\text{or} \qquad a_n \ge a_m > p - \varepsilon \quad \forall\, n \ge m \ \text{ or } \ a_n > p - \varepsilon \quad \forall \ n \ge m. \qquad \ldots(i)$$

$$\text{Also} \qquad p = \sup S \ \Rightarrow \ a_n \le p < p + \varepsilon \ \forall\, n. \qquad \ldots(ii)$$

$$\therefore \qquad p - \varepsilon < a_n < p + \varepsilon \quad \forall\, n \ge m, \text{ using } (i), (ii)$$

$$|a_n - p| < \varepsilon \quad \forall \quad n \ge m.$$

Hence $\langle a_n \rangle$ converges to $p$.

**Remarks**

1. From the sufficient condition, it follows that :

*Every monotonically increasing bounded above sequence converges.*

2. *Similarly, every monotonically decreasing bounded below sequence converges.*

3. In the sufficient condition, we have seen that

$$\lim a_n = p, \quad \text{where} \quad p = \sup \langle a_n \rangle. \text{ Thus}$$

*A monotonically increasing bounded above sequence converges to its supremum (l.u.b.).*

4. *Similarly, a monotonically decreasing bounded below sequence converges to its infimum (g.l.b.).*

**Theorem 19.** *A monotonically increasing sequence which is not bounded above, diverges to* $+\infty$.

**Proof.** Let $\langle a_n \rangle$ be a monotonically increasing sequence which is not bounded above. Let $G$ be any positive number, however large. Since $\langle a_n \rangle$ is not bounded above, there exists some $m \in \mathbf{N}$ such that

$$a_m > G. \qquad \ldots(i)$$

Since $\langle a_n \rangle$ is monotonically increasing, therefore

$$a_n \ge a_m \quad \forall \ n \ge m.$$

From $(i)$ and $(ii)$, $\quad a_n > G \quad \forall \ n \ge m.$

Hence $\langle a_n \rangle$ diverges to $+\infty$.

**Remark.** *Similarly, a monotonically decreasing sequence which is not bounded below, diverges to* $-\infty$.

**Ex. 1.** *Prove that if a monotonically sequence is bounded above, then it is convergent.* [D.U., Maths (H) 1997]

**Hint.** Prove the sufficient condition of Theorem 18.

**Ex. 2.** *Show that a monotonically increasing sequence either is convergent to its supremum or diverges to* $+\infty$. [D.U., B.Sc. (G) 1988]

**Hint.** Prove Theorem 18 (sufficient condition) and Theorem 19.

**Ex. 3.** *Prove that a necessary and sufficient condition for a monotonically increasing sequence to be convergent is that it is bounded above.*

**Hint.** If $\langle a_n \rangle$ is monotonically increasing and convergent, then $\langle a_n \rangle$ is bounded $\Rightarrow \langle a_n \rangle$ is bounded above. Conversely, let $\langle a_n \rangle$ be monotonically increasing and bounded above, then by the sufficient condition of Theorem 18, $\langle a_n \rangle$ is convergent.

**Ex. 4.** *Show that a monotonically decreasing sequence is convergent iff it is bounded below.*

**Hint.** Let $\langle b_n \rangle$ be a monotonically decreasing sequence. If $\langle b_n \rangle$ is convergent, then $\langle b_n \rangle$ is bounded $\Rightarrow \langle b_n \rangle$ is bounded below. Conversely, let $\langle b_n \rangle$ be bounded below. Let $a_n = -b_n \; \forall \; n$. Then $\langle a_n \rangle$ is monotonically increasing and bounded above and so by the sufficient condition of Theorem 18, $\langle a_n \rangle$ is convergent to $l$ (say). Hence $\langle b_n \rangle = \langle -a_n \rangle$ is convergent to $-l$.

**Ex. 5.** *Prove that a monotonically decreasing sequence of real numbers either converges to its infimum or diverges to $-\infty$.*

[D.U., Physics (H) 1998]

## EXAMPLES

**Example 69.** *Show that the sequence $\langle S_n \rangle$, where*

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} \text{ is convergent.}$$

[D.U., Physics (H) 1993]

**Solution.** We have

$$S_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}.$$

Now $\quad S_{n+1} - S_n = \dfrac{1}{2n+1} + \dfrac{1}{2n+2} - \dfrac{1}{n+1}$

$$= \frac{1}{2(n+1)(2n+1)} > 0 \quad \forall \quad n \in \mathbf{N}.$$

$\therefore \qquad S_{n+1} \geq S_n \quad \forall \quad n \in \mathbf{N}.$

Thus the sequence $\langle S_n \rangle$ is monotonically increasing.

Again $\quad S_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{n+n} < \dfrac{1}{n} + \dfrac{1}{n} + \ldots + \dfrac{1}{n} = \dfrac{n}{n} = 1.$

$$\left( \because \; n+1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \text{ etc.} \right)$$

$\therefore \quad S_n < 1 \quad \forall \; n \Rightarrow \langle S_n \rangle$ is bounded above.

Since the sequence $\langle S_n \rangle$ is bounded above and monotonically increasing, $\langle S_n \rangle$ is convergent.

**Example 70.** *Show that the sequence $\langle S_n \rangle$, where*

$$S_n = \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \text{ is convergent.}$$

**Solution.** $S_{n+1} - S_n = \dfrac{1}{(n+1)!} > 0 \; \forall \; n \in \mathbf{N}.$

$\therefore \qquad S_{n+1} \geq S_n \quad \forall \quad n \in \mathbf{N}.$

Thus the sequence $\langle S_n \rangle$ is monotonically increasing.

Again $\quad S_n = \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots + \dfrac{1}{n!}$

$$\leq 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} \qquad (\because \; n! \geq 2^{n-1} \; \forall \; n)$$

which is a G.P. with common ratio $= \frac{1}{2}$.

$$\therefore \quad S_n \leq \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}} < 2 \;\; \forall \;\; n \in \mathbf{N}$$

$\Rightarrow \qquad S_n < 2 \;\; \forall \; n \in \mathbf{N}.$

Thus the sequence $\langle S_n \rangle$ is bounded above.

Since the sequence $\langle S_n \rangle$ is bounded above and monotonically increasing, $\langle S_n \rangle$ is convergent.

**Example 71.** *Prove that the sequence $\langle a_n \rangle$ defined by the relation*

$$a_n = 1, \; a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!} \; (n \geq 2), \textit{ converges.}$$

[D.U., B.Sc. (G) 1998, 93 ; B.A. (P) 1998]

**Solution.** We have

$$a_{n+1} - a_n = \frac{1}{n!} > 0 \quad \forall \quad n \in \mathbf{N}.$$

$\therefore \quad a_{n+1} \geq a_n \; \forall \; n \in \mathbf{N}$ and so $\langle a_n \rangle$ is monotonically increasing.

Again $\quad a_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots + \dfrac{1}{(n-1)!}$

$$< 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-2}}\right) = 1 + \frac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}}$$

$$= 1 + 2\left(1 - \frac{1}{2^{n-1}}\right) = 3 - \frac{1}{2^{n-2}} < 3, \;\; \forall \; n \geq 2.$$

Also $\qquad a_1 = 1 < 3.$

$\therefore \qquad a_n < 3 \;\; \forall \; n \in \mathbf{N}.$

Thus $\langle a_n \rangle$ is a bounded above and monotonically increasing sequence and consequently, $\langle a_n \rangle$ converges.

**Example 72.** *Let $\langle a_n \rangle$ be a sequence defined by :*

$$a_1 = 1, \; a_{n+1} = \frac{3 + 2a_n}{2 + a_n}, \; n \geq 1.$$

*Show that $\langle a_n \rangle$ is convergent and find its limit.*

[D.U., Physics (H) 1997 : B.Sc. (G) 1992]

**Solution.** The given sequence is

$$\langle 1, \tfrac{5}{3}, \tfrac{19}{11}, \ldots \rangle, \text{ where } a_2 > a_1, \; a_3 > a_2 \text{ etc.}$$

Taking the limit as $n \to \infty$ and using (1), we get

$$l^2 = 7 + l \quad \Rightarrow \quad l^2 - l - 7 = 0.$$

Hence $l$ is a positive root of $x^2 - x - 7 = 0$.

**Example 74.** *Prove that the sequence* $\langle a_n \rangle$ *recursively defined by*

$$a_1 = \sqrt{5},\ a_{n+1} = \sqrt{5 + a_n},\ n \geq 1$$

*convergs to the positive root of the equation* $x^2 - x - 5 = 0$.

[D.U., Maths (H) 1992]

**Hint.** Similar to Example 73.

**Example 75.** *Show that the sequence* $\langle a_n \rangle$ *defined by*

$$a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2a_n} \quad \forall\ n \geq 2$$

[D.U., Maths (H) 1999]

*converges to* 2.

**Solution.** We have $a_1 = \sqrt{2},\ a_2 = \sqrt{2\sqrt{2}}$. Then $a_2 > a_1$.

Let us suppose that $a_{k+1} > a_k$.

$$\Rightarrow \quad \sqrt{2a_{k+1}} > \sqrt{2a_k} \quad \Rightarrow \quad a_{k+2} > a_{k+1}.$$

So, by the principle of induction, $a_{n+1} > a_n\ \forall\ n \in \mathbf{N}$.

Thus the sequence $\langle a_n \rangle$ is monotonically increasing.

Again we have $a_1 = \sqrt{2} < 2$.

Let $\quad a_k < 2 \quad \Rightarrow \quad \sqrt{2a_k} < \sqrt{2 \, . \, 2} = 2 \ \Rightarrow a_{k+1} < 2.$

By the principle of induction, $a_n < 2\ \forall\ n \Rightarrow \langle a_n \rangle$ is bounded above.

Since $\langle a_n \rangle$ is monotonically increasing and bounded above, therefore, $\langle a_n \rangle$ is convergent.

Let $\quad \lim\limits_{n \to \infty} a_n = l \quad \Rightarrow \quad \lim\limits_{n \to \infty} a_{n+1} = l.$

Now $\quad a_{n+1} = \sqrt{2a_n} \quad \Rightarrow \quad a_{n+1}^2 = 2a_n$. Taking the limit a $n \to \infty$,

$$l^2 = 2l \quad \Rightarrow \quad l^2 - 2l = 0 \quad \Rightarrow \quad l = 0,\ l = 2.$$

Hence $\lim\limits_{n \to \infty} a_n = 2$, since $\langle a_n \rangle$ converges to its *l.u.b.* 2.

**Example 76.** *Show that the sequence* $\langle S_n \rangle$ *defined by the formula* $S_{n+1} = \sqrt{3S_n}$ , $S_1 = 1$ *converges to* 3.

**Hint.** Similar to Example 75.

**Example 77.** *Let* $a_1 = 1,\ a_{n+1} = \sqrt{2 + a_n}$, *whenever* $n \geq 1$. *Show that the sequence* $\langle a_n \rangle$ *is convergent. Also find its limit.* [D.U., B.A. (P) 1992]

**Solution.** The given sequence is $\langle 1, \sqrt{3}, \sqrt{2 + \sqrt{3}}, \dots \rangle$.

Obviously $a_2 > a_1$. Let us suppose that $a_{k+1} > a_k \Rightarrow 2 + a_{k+1} > 2 + a_k$

$$\Rightarrow \quad \sqrt{2 + a_{k+1}} > \sqrt{2 + a_k} \quad \Rightarrow a_{k+2} > a_{k+1}.$$

By the principle of induction, $a_{n+1} > a_n\ \ \forall\ n$ and so $\langle a_n \rangle$ is monotonically increasing.

Again $a_{n+1} > a_n \Rightarrow \sqrt{2+a_n} > a_n$

$\Rightarrow 2 + a_n - a_n^2 > 0 \Rightarrow (2 - a_n)(1 + a_n) > 0 \Rightarrow 2 - a_n > 0$

$(\because 1 + a_n > 0)$

$\Rightarrow a_n < 2 \; \forall \; n \in \mathbf{N} \Rightarrow \langle a_n \rangle$ is bounded above by 2.

Since $\langle a_n \rangle$ is monotonically increasing and bounded above, therefore, $\langle a_n \rangle$ is convergent.

Let $\lim\limits_{n \to \infty} a_n = l$, so that $\lim\limits_{n \to \infty} a_{n+1} = l$.

Now $a_{n+1} = \sqrt{2+a_n} \Rightarrow \lim\limits_{n \to \infty} a_{n+1}^2 = \lim\limits_{n \to \infty} (2 + a_n)$

$\Rightarrow \quad l^2 = 2 + l \Rightarrow l^2 - l - 2 = 0 \Rightarrow (l+1)(l-2) = 0$

$\Rightarrow \quad l = -1$ or $l = 2$.

Since all the terms of the sequence $\langle a_n \rangle$ are positive, so $l > 0$. Hence $\lim\limits_{n \to \infty} a_n = 2$.

**Example 78.** *A sequence $\langle a_n \rangle$ is defined as follows :*

$$a_1 = 1, \quad a_{n+1} = \frac{4 + 3a_n}{3 + 2a_n}, \; n \geq 1.$$

*Show that $\langle a_n \rangle$ converges and find its limit.*

[D.U., Maths (H) 1993 ; B.A. (P) 1994]

**Solution.** The given sequence is $< 1, \frac{7}{5}, \frac{41}{29}, \ldots >$.

We have $a_1 = 1 < 2$. Now

$$a_{n+1} = 2 - \frac{2 + a_n}{3 + 2a_n} < 2 \; \forall \; n \in \mathbf{N}. \qquad (\because a_n > 0)$$

Thus the sequence $\langle a_n \rangle$ is bounded above.

Obviously $a_2 > a_1$. ...(1)

Let us suppose that $a_{k+1} > a_k \Rightarrow a_{k+1} - a_k > 0$.

Now $a_{k+2} - a_{k+1} = \left( \dfrac{4 + 3a_{k+1}}{3 + 2a_{k+1}} - \dfrac{4 + 3a_k}{3 + 2a_k} \right)$

$$= \frac{a_{k+1} - a_k}{(3 + 2a_{k+1})(3 + 2a_k)} > 0, \text{ by (1)}$$

$\therefore \quad a_{k+2} > a_{k+1}$.

Thus, by the principle of induction, $a_{n+1} > a_n \; \forall \; n$ and so $\langle a_n \rangle$ is a monotonically increasing sequence. Since $\langle a_n \rangle$ is a monotonically increasing and bounded above sequence, therefore, $\langle a_n \rangle$ is convergent.

Let $\lim\limits_{n \to \infty} a_n = l$, so that $\lim\limits_{n \to \infty} a_{n+1} = l$.

Now $a_{n+1} = \dfrac{4 + 3a_n}{3 + 2a_n}$

$\Rightarrow \lim\limits_{n \to \infty} (3a_{n+1} + 2a_{n+1} a_n) = \lim\limits_{n \to \infty} (4 + 3a_n)$

$\Rightarrow 3l + 2l^2 = 4 + 3l \Rightarrow l^2 = 2 \Rightarrow l = \sqrt{2}$, since $a_n > 0 \; \forall \; n$.

Hence $\lim\limits_{n \to \infty} a_n = \sqrt{2}$.

**Example 79.** *Let $\langle a_n \rangle$ be a sequence defined as follows :*

$$a_1 = \frac{3}{2}, \; a_{n+1} = 2 - \frac{1}{a_n}, \; \forall \; n \geq 1.$$

*Show that $\langle a_n \rangle$ is monotonic and bounded and converges to 1.*

[D.U., Maths (H) 1994, 95 ; Physics (H) 1998]

**Solution.** We have $a_1 = \dfrac{3}{2}, \; a_2 = 2 - \dfrac{1}{a_1} = 2 - \dfrac{2}{3} = \dfrac{4}{3}$.

$\therefore \; a_2 < a_1$. Let $a_{k+1} < a_k$. Then $-a_{k+1} > -a_k$

$\Rightarrow -\dfrac{1}{a_{k+1}} < -\dfrac{1}{a_k} \Rightarrow 2 - \dfrac{1}{a_{k+1}} < 2 - \dfrac{1}{a_k} \Rightarrow a_{k+2} < a_{k+1}$.

By induction, $\langle a_n \rangle$ is a monotonically decreasing sequence.

Obviously, $1 < a_1 = \dfrac{3}{2}$.

Let us suppose that $1 < a_k \Rightarrow a_k > 1 \Rightarrow a_k - 1 > 0$.

We have $a_{k+1} - 1 = 2 - \dfrac{1}{a_k} - 1 \Rightarrow \dfrac{a_k - 1}{a_k} > 0$

$\Rightarrow \quad a_{k+1} - 1 > 0 \Rightarrow 1 < a_{k+1}$. Thus $1 \leq a_n \; \forall \; n \in \mathbf{N}$.

So $\langle a_n \rangle$ is bounded below. Since $\langle a_n \rangle$ is monotonically decreasing and bounded below, therefore $\langle a_n \rangle$ is convergent.

Let $\lim\limits_{n \to \infty} a_n = l$, so that $\lim\limits_{n \to \infty} a_{n+1} = l$.

Now $a_{n+1} = 2 - \dfrac{2}{a_n} \Rightarrow \lim a_{n+1} = 2 - \dfrac{1}{\lim a_n}$

$\Rightarrow l = 2 - \dfrac{1}{l} \Rightarrow l^2 - 2l + 1 = 0$

$\Rightarrow (l-1)^2 = 0 \Rightarrow l - 1 = 0$. Hence $l = 1$.

**Example 80.** *A sequence $\langle a_n \rangle$ is defined as follows :*

$$a_1 > 0, \; a_{n+1} = \frac{1}{2}\left( a_n + \frac{2}{a_n} \right), \; n \geq 1. \qquad \text{...(1)}$$

*Show that $\langle a_n \rangle$ is a bounded monotonic sequence. Find its limit.*

Thus $\langle t_n \rangle$ is a monotonically decreasing sequence.

Since $\langle t_n \rangle$ is a monotonically decreasing and bounded below sequence, it is convergent.

Let $\lim\limits_{n \to \infty} t_n = l$, so that $\lim\limits_{n \to \infty} t_{n+1} = l$. ...(2)

Taking the limit as $n \to \infty$ on both sides of (1) and using (2), we get

$$l = \frac{1}{2}\left(l + \frac{9}{l}\right) \Rightarrow l^2 = 9 \Rightarrow l = 3,$$

since $\langle t_n \rangle$ is a sequence of positive terms.

Hence $\langle t_n \rangle$ converges to 3.

**Example 82.** *A sequence* $\langle s_n \rangle$ *is defined by*

$$s_1 = 1,\ s_{n+1} = \left(\frac{3 + s_n^2}{2}\right)^{1/2}$$

*Show that* $\langle s_n \rangle$ *is a bounded monotonically increasing sequence and converges to* $\sqrt{3}$. [D.U.; Maths (H) 1996]

**Solution.** We have

$$s_2 = \left(\frac{3 + s_1^2}{2}\right)^{1/2} = \left(\frac{3 + 1}{2}\right)^{1/2} = \sqrt{2}.\ \text{Thus } s_2 > s_1.$$

Let $s_{k+1} > s_k$ so that $3 + s_{k+1}^2 > 3 + s_k^2$

$$\Rightarrow \quad \left(\frac{3 + s_{k+1}^2}{2}\right)^{1/2} > \left(\frac{3 + s_k^2}{2}\right)^{1/2} \Rightarrow s_{k+2} > s_{k+1}.$$

Thus, by the principle of induction, $s_{n+1} > s_n$ for each $n$ and so $\langle s_n \rangle$ is a monotonically increasing sequence.

Now we show that $\langle s_n \rangle$ is bounded.

We have $s_1 = 1 < \sqrt{3}$. Let $s_k \le \sqrt{3}$, so that

$$\sqrt{\frac{3 + s_k^2}{2}} \le \sqrt{\frac{3 + 3}{2}} = \sqrt{3} \Rightarrow s_{k+1} \le \sqrt{3}.$$

By the principle of induction, $s_n \le \sqrt{3}$ for each $n$.

Obviously, $s_n \ge 1$ for each $n$ and so $1 \le s_n \le \sqrt{3}$ for each $n$.

It follows that $\langle s_n \rangle$ is bounded. Since $\langle s_n \rangle$ is a bounded and monotonically increasing sequence, $\langle s_n \rangle$ is convergent.

Let $\lim\limits_{n \to \infty} s_n = l$, so that $\lim\limits_{n \to \infty} s_{n+1} = l$

$$\text{Now } s_{n+1} = \left(\frac{3 + s_n^2}{2}\right)^{1/2} \Rightarrow 2 s_{n+1}^2 = 3 + s_n^2.$$

Taking the limit on both sides as $n \to \infty$, we obtain

$$2l^2 = 3 + l^2 \Rightarrow l^2 = 3 \Rightarrow l = \sqrt{3},$$

since $\langle s_n \rangle$ is a sequence of positive terms.

Hence $\langle s_n \rangle$ converges to $\sqrt{3}$.

**Example 83.** *Let $\langle a_n \rangle$ be a sequence of real numbers defined as*

$$a_1 = 1,\ a_{n+1}^2 = \frac{2 + a_n^2}{2} \text{ for } n \geq 1.$$

*Show that $\langle a_n \rangle$ is a monotonically increasing sequence converging to $\sqrt{2}$.* [D.U., B.C.A. 1999]

**Hint.** Similar to Example 82.

**Example 84.** *Let $\langle a_n \rangle$ be defined as*

$$a_1 = 1,\ a_{n+1}^2 = \frac{4 + a_n^2}{4} \text{ for } n \geq 1.$$

*Show that $\langle a_n \rangle$ is a monotonically increasing sequence converging to 2.*

**Please try yourself.**

**Example 85.** *Let $s_1 = 1$ and $s_{n+1} = \dfrac{s_n + 1}{3}\ \forall\ n \geq 1.$* ...(1)

(*i*) *Find $s_2, s_3, s_4$.*

(*ii*) *Use induction to show that $s_n > \frac{1}{2}\ \forall\ n$.*

(*iii*) *Show that $\langle s_n \rangle$ is a monotonically decreasing sequence.*

(*iv*) *Show that $\lim s_n$ exists and find $\lim s_n$.* [D.U., Maths (H) 1997]

**Solution.** From (1), we have

$$s_2 = \frac{s_1 + 1}{3} = \frac{1 + 1}{3} = \frac{2}{3},$$

$$s_3 = \frac{s_2 + 1}{3} = \frac{\frac{2}{3} + 1}{3} = \frac{5}{9},\quad s_4 = \frac{s_3 + 1}{3} = \frac{\frac{5}{9} + 1}{3} = \frac{14}{27}.$$

(*ii*) We see that $s_1 > \frac{1}{2}$, $s_2 > \frac{1}{2}$ etc. Let $s_k > \frac{1}{2}$. Then

$$s_{k+1} = \frac{s_k + 1}{3} > \frac{\frac{1}{2} + 1}{3} = \frac{1}{2} \text{ and so } s_n > \frac{1}{2}\ \forall\ n.$$

(*iii*) We see that $s_2 < s_1$, $s_3 < s_2$ etc. Let $s_{k+1} < s_k$. Then

$$s_{k+2} - s_{k+1} = \frac{s_{k+1} + 1}{3} - \frac{s_k + 1}{3} = \frac{s_{k+1} - s_k}{3} < 0.$$

$\therefore$ $s_{k+2} < s_{k+1}$. By the principle of induction, $s_{n+1} < s_n\ \forall\ n$ and so $\langle s_n \rangle$ is monotonically decreasing. Also $\langle s_n \rangle$ is bouded below. It follows that $\langle s_n \rangle$ is convergent. Let $\lim s_n = l = \lim s_{n+1}$. From (1), we get

$$3 \lim s_{n+1} = \lim s_n + 1 \text{ or } 3l = l + 1. \quad \text{Hence } l = \lim s_n = \frac{1}{2}.$$

**Example 86.** *Show that the sequence $\langle a_n \rangle$ defined by*

$$a_{n+1} = 1 - \sqrt{1 - a_n}\ \forall\ n \geq 1 \text{ and } a_1 < 1$$

*verges to 0.* [D.U., Physics (H) 1995]

**Solution.** We have $a_1 < 1$.

Let $a_k < 1$ so that $1 - a_k > 0$. $\Rightarrow 1 - \sqrt{1 - a_k} < 1 \Rightarrow a_{k+1} < 1$.

By the principle of induction, $a_n < 1\ \forall\ n \in \mathbf{N}$. ...(1)

Thus $\langle a_n \rangle$ is bounded below.

We have $$a_{n+1} = \frac{(1 - \sqrt{1 - a_n})(1 + \sqrt{1 - a_n})}{1 + \sqrt{1 - a_n}}$$

$$a_{n+1} = \frac{1 - (1 - a_n)}{1 + \sqrt{1 - a_n}} = \frac{a_n}{1 + \sqrt{1 - a_n}}. \quad \text{...(2)}$$

From (1), $a_n < 1 \Rightarrow 1 - a_n > 0 \Rightarrow 1 + \sqrt{1 - a_n} > 1$

$$\Rightarrow \frac{a_n}{1 + \sqrt{1 - a_n}} < a_n \Rightarrow a_{n+1} < a_n, \text{ by (2).}$$

$\therefore$ $a_{n+1} < a_n\ \forall\ n.$

Thus $\langle a_n \rangle$ is monotonically decreasing and bounded below and so $a_n \rangle$ is convergent.

Let $\lim\limits_{n \to \infty} a_n = l$, so that $\lim\limits_{n \to \infty} a_{n+1} = l$. ...(3)

It is clear that $l < 1$, using (1).

We have $\sqrt{1 - a_n} = 1 - a_{n+1}$

$\Rightarrow 1 - a_n = (1 - a_{n+1})^2 = 1 - 2a_{n+1} + a_{n+1}^2$

$\Rightarrow a_{n+1}^2 - 2a_{n+1} + a_n = 0.$

Taking the limit as $n \to \infty$ and using (3), we obtain

$$l^2 - 2l + l = 0 \Rightarrow l(l - 1) = 0 \Rightarrow l = 0, \text{ since } l < 1.$$

Hence $\lim\limits_{n \to \infty} a_n = 0.$

**Example 87.** *If $\langle s_n \rangle$ is a bounded sequence such that*

$$s_{n+1} = \sqrt{\frac{ab^2 + s_n^2}{a + 1}},\ b > a\ \forall\ n \geq 1, s_1 = a > 0,$$

*hen show that the sequence $\langle s_n \rangle$ is an increasing sequence and $\lim s_n = b$.* [D.U., Maths (H) 1990]

**Solution.** We have

$$s_2^2 - s_1^2 = \frac{ab^2 + a^2}{a + 1} - a^2 = \frac{a(b^2 - a^2)}{a + 1} > 0. \quad (\because a > 0 \text{ and } b > a)$$

$(\because s_2 > 0, s_1 > 0)$

$\therefore$ $s_2^2 > s_1^2 \Rightarrow s_2 > s_1.$

$\therefore$ $s_{n+1} > s_n$ so that $s_{n+1}^2 > s_n^2 \Rightarrow ab^2 + s_{n+1}^2 > ab^2 + s_n^2$

$$\Rightarrow \quad \sqrt{\frac{ab^2 + s_{n+1}^2}{a+1}} > \sqrt{\frac{ab^2 + s_n^2}{a+1}} \qquad (\because a > 0)$$

$$\Rightarrow \quad s_{n+2} > s_{n+1}.$$

Thus, by the principle of induction, $s_{n+1} > s_n \; \forall \; n$ and so $\langle s_n \rangle$ is a monotonically increasing sequence.

Now we proceed to show that $\langle s_n \rangle$ is bounded above.

We have $s_1 = a < b$. $\qquad (\because s_n > 0$ and $b > 0)$

Let $s_n < b$ so that $s_n^2 < b^2$.

$$\Rightarrow \quad ab^2 + s_n^2 < ab^2 + b^2 = b^2 (a+1)$$

$$\Rightarrow \quad \sqrt{\frac{ab^2 + s_n^2}{a+1}} < \sqrt{\frac{b^2 (a+1)}{a+1}} = b \qquad (\because a + 1 > 0)$$

$$\Rightarrow \quad s_{n+1} < b.$$

Thus, by the principle of induction, $s_n < b \; \forall \; n$ and so $\langle s_n \rangle$ is bounded above. Since $\langle s_n \rangle$ is monotonically increasing bounded above, $\langle s_n \rangle$ is convergent.

Let $\lim\limits_{n \to \infty} s_n = l$, so that $\lim\limits_{n \to \infty} s_{n+1} = l$.

We have $s_{n+1}^2 = \dfrac{ab^2 + s_n^2}{a+1}$.

Proceeding to the limit as $n \to \infty$ and using (1), we obtain

$$(a+1)\, l^2 = ab^2 + l^2 \quad \Rightarrow \quad l^2 = b^2 \quad \Rightarrow \quad l = b, \text{ since } l > 0.$$

Hence $\lim\limits_{n \to \infty} s_n = b$.

**Example 88.** *Show that the sequence* $\langle S_n \rangle$, *where* $S_n = \left(1 + \dfrac{1}{n}\right)^n$ *is convergent and that* $\lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$ *lies between* 2 *and* 3.

[D.U., B.Sc. (G) 1996 ; Physics (H) 1996, 94 ; B.A. (P) 1990]

**Solution.** By Binomial Theorem, we have

$$S_n = \left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \dots + \frac{n(n-1)(n-2) \dots. 1}{n!}\left(\frac{1}{n}\right)^n.$$

or

$$S_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{n-1}{n}\right). \qquad \dots(1)$$

Similarly,

$$S_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \dots + \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \dots + \left(1 - \frac{1}{n+1}\right). \qquad \dots(2)$$

Now $\; n + 1 > n \;\; \Rightarrow \; -(n+1) < -n \;\; \Rightarrow \; -\dfrac{1}{n+1} > -\dfrac{1}{n}$

$$\Rightarrow \quad \left(1 - \frac{1}{n+1}\right) > \left(1 - \frac{1}{n}\right) \text{ and } \left(1 - \frac{2}{n+1}\right) > \left(1 - \frac{2}{n}\right) \text{etc.}$$

From (1) and (2), we see that

$S_{n+1} > S_n \; \forall \; n$. Thus $\langle S_n \rangle$ is monotonically increasing.

We shall now prove that $\langle s_n \rangle$ is bounded above. We have

$$S_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{n-1}{n}\right)$$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!}$$

$$\le \underbrace{1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}} \qquad (\because n! \ge 2^{n-1} \; \forall \; n)$$

$$= 1 + \left(\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}\right) = 1 + 2\left(1 - \frac{1}{2^n}\right) = 3 - \frac{1}{2^{n-1}}.$$

$\therefore \qquad S_n < 3 \; \forall \; n \in \mathbf{N}$ and thus $\langle S_n \rangle$ is bounded above.

Since $\langle S_n \rangle$ is bounded above and monotonically increasing, therefore, $\langle s_n \rangle$ is a convergent sequence.

From (1), $\quad S_n > 2 \; \forall \; n$. Thus $2 < S_n < 3 \; \forall \; n \in \mathbf{N}$

$$\Rightarrow \quad 2 < \lim_{n \to \infty} S_n < 3.$$

Hence $\lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$ lies between 2 and 3.

**Remark.** Indeed, it can be proved that

$$\lim_{\mathbf{n} \to \infty} \left(1 + \frac{1}{\mathbf{n}}\right)^{\mathbf{n}} = \mathbf{e}, \textbf{ where } \mathbf{e} = \mathbf{2.7182\dots}$$

**Example 89.** *If* $a_n = \left(1+\frac{1}{n}\right)^{n+1}$, *then show that* $\langle a_n \rangle$ *converges to* $e$.

**Solution.** By Example 88, $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

Now
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)$$
$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n . \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = e\,.\,(1+0) = e.$$

Hence $\lim\limits_{n\to\infty} a_n = e.$

**Example 90.** *Prove that*

$$(a)\ \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2, \qquad (b)\ \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = e.$$

**Solution.** (*a*) Let $n = 2m$ so that $n\to\infty \Leftrightarrow m\to\infty$.

$$\therefore\quad \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{2m}$$
$$= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m \times \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e\times e = e^2.$$

(*b*) Let $n+1 = m$ so that $n\to\infty \Leftrightarrow m\to\infty$.

$$\therefore\quad \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m-1}$$
$$= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m \times \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{-1} = e\,.\,1 = e.$$

## EXERCISES

1. Prove that the sequence $\langle a_n \rangle$ defined by
$$a_1 = \sqrt{6} \text{ and } a_{n+1} = \sqrt{6+a_n} \text{ for } n \ge 1$$
converges to 3.
2. Let $a_1 > 0$ and $a_{n+1} = \sqrt{6+a_n}$ for $n \ge 1$. Show that the sequence $\langle a_n \rangle$ is a monotone increasing sequence if $0 < a_1 < 3$ and a monotone decreasing sequence if $a_1 > 3$. Find $\lim\limits_{n\to\infty} a_n$.
3. Prove that the sequence $\langle a_n \rangle$ defined by
$$a_1 = a > 0 \text{ and } a_{n+1} = \frac{2a_n}{a_n+1}$$
converges to 1.

4. Let $a_1 = \alpha$, $a_{n+1} = \frac{1}{2}\left(a_n^2 + a_n\right)$ for $n \ge 1$. Show that $\langle a_n \rangle$ converges to 1 if $0 \le \alpha < 1$. What happens if $\alpha = 1$ ?

   [**Hint.** $\langle a_n \rangle = \langle 1, 1, 1, \dots \rangle$, if $\alpha = 1$. Hence $a_n \to 1$, if $\alpha = 1$.]
5. Let $s_n = \dfrac{1.3.5 \dots (2n-1)}{2.4.6 \dots 2n}$ $\forall\ n \in \mathbf{N}$. Prove that $\langle s_n \rangle$ is convergent and $\lim\limits_{n\to\infty} s_n \le \frac{1}{2}$.

   [**Hint.** $s_1 = \frac{1}{2}$, $s_2 = \frac{1.3}{2.4}$, $s_3 = \frac{1.3.5}{2.4.6}$ etc. $\Rightarrow s_1 > s_2 > s_3$ etc. $\Rightarrow \langle s_n \rangle$ is a monotonically decreasing sequence, bounded below by 0. Hence $\langle s_n \rangle$ is convergent. Since $s_n \le \frac{1}{2}\ \forall\ n \in \mathbf{N}$, $\lim\limits_{n\to\infty} s_n \le \frac{1}{2}$.]

### 3.11 Subsequence

Roughly speaking, a *subsequence* of a sequence $\langle a_n \rangle$ is a subset of $\langle a_n \rangle$, which is also a sequence in its own right. For example

$$\langle 1, 3, 5, \dots, 2n-1, \dots \rangle, \langle 2, 4, 6, \dots, 2n, \dots \rangle$$

are subsequences of the sequence $\langle 1, 2, 3, 4, \dots, n, \dots \rangle$.

It must, however, be noted that the order of the terms of a subsequence must be the same as that of the given sequence. Thus

$$\langle 3, 5, 1, 7, \dots \rangle \text{ is not a subsequence of } \langle 1, 2, 3, 4, \dots, n, \dots \rangle.$$

Now we give the proper definition of a *subsequence*.

**Definition.** *Let* $\langle a_n \rangle$ *be any sequence. Let* $\langle n_1, n_2, \dots, n_k \dots \rangle$ *be a sequence of positive integers such that* $i > j \Rightarrow n_i > n_j$. *Then the sequence* $\langle a_{n_1}, a_{n_2}, \dots, a_{n_k}, \dots \rangle$, *written as* $\langle a_{n_k} \rangle$, *is called a* **subsequence** *of* $\langle a_n \rangle$.

**Remark.** The condition $i > j \Rightarrow n_i > n_j$ guarantees that the order of the terms of a subsequence is the same as that of the given sequence.

**Illustrations**

1. $\langle 1, 9, 25, \dots, (2n-1)^2, \dots \rangle$ is a subsequence of $\langle n^2 \rangle$.
2. $\langle a_{10}, a_{11}, a_{12}, \dots \rangle$ is a subsequence of $\langle a_n \rangle$.

**Theorem 20.** *If a sequence* $\langle a_n \rangle$ *converges to* $l$, *then every subsequence of* $\langle a_n \rangle$ *converges to* $l$.

**Proof.** Let $\langle a_{n_k} \rangle$ be a subsequence of $\langle a_n \rangle$.

Since $a_n \to l$, therefore, for any $\varepsilon > 0$, there exists a positive integer $m$ such that

$$|a_n - l| < \varepsilon \quad \forall \quad n \ge m.$$

In particular,
$$|a_{n_k} - l| < \varepsilon\ \forall\ n_k \ge m.$$

Hence $\langle a_{n_k} \rangle$ converges to $l$.

**Remark.** *The converse of the above theorem may not be true.*

For example, the sequence $\langle (-1)^n \rangle = \langle -1, 1, -1, 1, \dots \rangle$ does not converge, but its subsequence $\langle 1, 1, 1, \dots \rangle$ converges.

We recall that a real number $p$ is a *limit point* of a sequence $\langle a_n \rangle$, if for each $\varepsilon > 0$,

$$a_n \in \,]p-\varepsilon, p+\varepsilon[ \text{ for infinitely many values of } n.$$

The following equivalent definition of limit point is an easy consequence of the above definition :

*A real number p is a* **limit point** *or* **cluster point** *of a sequence* $\langle a_n \rangle$, *if for each* $\varepsilon > 0$ *and each positive integer m, there exists a positive integer* $k > m$ *such that* $a_k \in \,]p-\varepsilon, p+\varepsilon[$

i.e., $$|a_k - p| < \varepsilon \text{ for } k > m.$$

**Theorem 21.** *A real number l is a limit point (or cluster point) of a sequence* $\langle a_n \rangle$ *if and only if there exists a subsequence of* $\langle a_n \rangle$ *converging to l.* [D.U., Maths (H) 1999, 92]

**Proof.** *The condition is sufficient.*

Suppose there exists a subsequence $\langle a_{n_k} \rangle$ of the given sequence $\langle a_n \rangle$ such that $a_{n_k} \to l$.

We shall prove that $l$ is a limit point of $\langle a_n \rangle$.

Let $\varepsilon > 0$ be any positive number.

Since $a_{n_k} \to l$, therefore, there exists a positive integer $j$ such that

$$|a_{n_k} - l| < \varepsilon \quad \forall \quad k \geq j.$$

i.e., $$a_{n_k} \in \,]l-\varepsilon, l+\varepsilon[ \text{ for infinitely many values of } k.$$

Since $\langle a_{n_k} \rangle$ is a subsequence of $\langle a_n \rangle$, so any *nbd.* $]l-\varepsilon, l+\varepsilon[$ of $l$ contains infinitely many members of $\langle a_n \rangle$ and thus $l$ is a limit point of $\langle a_n \rangle$.

*The condition is necessary.*

Let $l$ be a limit point of the sequence $\langle a_n \rangle$.

We shall find a subsequence of $\langle a_n \rangle$ which converges to $l$.

Since $l$ is a limit point of $\langle a_n \rangle$, for $\varepsilon = 1$ and $m = 1$, there exists a positive integer $n_1 > 1$, such that

$$|a_{n_1} - l| < 1.$$

Again for $\varepsilon = \frac{1}{2}$ and $m = n_1$, there exists a positive integer $n_2 > n_1$, such that

$$|a_{n_2} - l| < \frac{1}{2}.$$

Similarly, by taking $\varepsilon = \frac{1}{3}$ and $m = n_2$, there exists a positive integer $n_3 > n_2$, such that

$$|a_{n_3} - l| < \frac{1}{3}.$$

Proceeding in the same manner, we obtain

$$|a_{n_k} - l| < \frac{1}{k}. \qquad \dots(1)$$

After obtaining $a_{n_1}, a_{n_2}, \dots, a_{n_k}$, if we take $\varepsilon = \frac{1}{k+1}$ and $m = n_k$, then we can find a positive integer $n_{k+1} > n_k$ such that

$$|a_{n_{k+1}} - l| < \frac{1}{k+1}.$$

Thus by induction, we get a subsequence $\langle a_{n_k} \rangle$ of $\langle a_n \rangle$.

We shall now show that $\langle a_{n_k} \rangle$ converges to $l$.

Given any $\varepsilon > 0$, we can find a positive integer $j$ such that

$$\frac{1}{j} < \varepsilon. \qquad \dots(2)$$

From (1) and (2), $$|a_{n_k} - l| < \frac{1}{k} \leq \frac{1}{j} < \varepsilon, \ \forall \ k \geq j.$$

$\therefore$ $$|a_{n_k} - l| < \varepsilon, \quad \forall \quad k \geq j.$$

Hence $\langle a_{n_k} \rangle$ converges to $l$.

**Corollary.** *Every bounded sequence has a convergent subsequence.*

**Proof.** Let $\langle a_n \rangle$ be a bounded sequence. Then $\langle a_n \rangle$ must have a limit point say $l$ [By Bolzano-Weierstrass Theorem]. By the above theorem, $\langle a_n \rangle$ has a subsequence $\langle a_{n_k} \rangle \to l$.

## EXAMPLES

**Example 91.** *Find all limit points of the sequence* $\langle 2 + (-1)^n \rangle$. ***Also*** *find subsequences converging to these limit points.* [B.C.A. Sample Problem]

**Solution.** We have

$$\langle 2 + (-1)^n \rangle = \langle 1, 3, 1, 3, 1, 3, \dots \rangle$$

It has two limit points *viz.* 1 and 3. The given sequence has two subsequences :

$$\langle a_{2n-1} \rangle = \langle 1, 1, 1, \dots \rangle \to 1$$

and $$\langle a_{2n} \rangle = \langle 3, 3, 3, \dots \rangle \to 3.$$

**Example 92.** *Find all limit points of the sequences :*

(*a*) $\langle (-1)^n \rangle$, (*b*) $\langle 1 + (-1)^n \rangle$.

*Also find subsequences converging to these limit points.*

**Please try yourself.**

**Example 93.** *If* $a_n \to +\infty$, *show that every subsequence of* $\langle a_n \rangle$ *also diverges to* $+\infty$.

**Solution.** Since $a_n \to +\infty$, therefore, given any number $G > 0$, however large, there exists a positive integer $m$ such that

$$a_n > G \quad \text{when} \quad n \geq m. \qquad \dots(1)$$

If $\langle a_{n_k} \rangle$ is any subsequences of $\langle a_n \rangle$, then by (1)

$$a_{n_k} > G \quad \text{when} \quad n_k \geq m.$$

Hence $a_{n_k} \to +\infty$.

**Example 94.** *If $a_n \to -\infty$, show that every subsequence of $\langle a_n \rangle$ also diverges to $-\infty$.*

**Solution.** Since $a_n \to -\infty$, therefore, given any negative number $g$, however small, there exists a positive integer $m$ such that

$$a_n < g \quad \text{when} \quad n \geq m. \qquad \text{...(1)}$$

If $\langle a_{n_k} \rangle$ is any subsequence of $\langle a_n \rangle$, then by (1)

$$a_{n_k} < g \quad \text{when} \quad n_k \geq m.$$

Hence $a_{n_k} \to -\infty$.

**Example 95.** *Show that a monotone sequence can have at most one cluster point.* [D.U., Maths (H) 1996]

**Solution.** Let $\langle a_n \rangle$ be a *monotonically increasing* sequence. We have to show that either $\langle a_n \rangle$ has no cluster point or exactly one cluster point.

**Case I.** Let $\langle a_n \rangle$ *be bounded above.* Since $\langle a_n \rangle$ is monotonically increasing and bounded above, so $\langle a_n \rangle$ is convergent (Theorem 18) and hence $\langle a_n \rangle$ has exactly one cluster point (Theorem 11).

**Case II.** Let $\langle a_n \rangle$ *be not bounded above.* Then $\langle a_n \rangle$ diverges to $+\infty$ (Theorem 19). Let, if possible, $\langle a_n \rangle$ have a cluster point $l$. Then by Theorem 21, there exists a subsequence $\langle a_{n_k} \rangle$ converging to $l$ and $a_n \to +\infty$. This is impossible (See Example 93). Hence $\langle a_n \rangle$ has no cluster point.

**Example 96.** *If each of the two subsequences $\langle a_{2n-1} \rangle$ and $\langle a_{2n} \rangle$ of a equence $\langle a_n \rangle$ converges to $l$, then show that the sequence $\langle a_n \rangle$ also con-rges to $l$.*

**Solution.** We have

$$\langle a_{2n-1} \rangle = \langle a_1, a_3, a_5, \dots \rangle, \ \langle a_{2n} \rangle = \langle a_2, a_4, a_6, \dots \rangle.$$

Since $a_{2n-1} \to l$ and $a_{2n} \to l$, therefore, for any $\varepsilon > 0$, there exist positive tegers $m_1$ and $m_2$ such that

$$|a_{2n-1} - l| < \varepsilon \quad \forall \quad n \geq m_1, \qquad \text{...(1)}$$

$$|a_{2n} - l| < \varepsilon \quad \forall \ n \geq m_2. \qquad \text{...(2)}$$

Let $m = \max(m_1, m_2)$, so that $m \geq m_1$ and $m \geq m_2$.

Then (1) and (2) are both satisfied for $n \geq m$. Further

$$|a_n - l| < \varepsilon \ \forall \ n \geq m.$$

Hence the sequence $\langle a_n \rangle$ converges to $l$.

**Example 97.** *Given that $\langle a_n \rangle$ is a sequence such that*

$$a_2 \leq a_4 \leq a_6 \leq \dots \leq a_5 \leq a_3 \leq a_1 \qquad \text{...(1)}$$

*and a sequence $\langle b_n \rangle$ where $b_n = a_{2n-1} - a_{2n}$, converges to 0 ; then show that the sequence $\langle a_n \rangle$ is convergent.*

**Solution.** Clearly

$$\langle a_2, a_4, a_6, \dots, a_{2n}, \dots \rangle, \langle a_1, a_3, a_5, \dots, a_{2n-1}, \dots \rangle$$

are two subsequences of the sequence $\langle a_n \rangle$.

From (1), $a_2 \leq a_4 \leq a_6 \leq a_8 \leq \dots$ and so the subsequence $\langle a_{2n} \rangle$ is monotonically increasing and bounded above by $a_1$.

Thus $\langle a_{2n} \rangle$ is convergent. Let $\lim\limits_{n \to \infty} a_{2n} = l$. ...(2)

Again from (1), $\dots a_7 \leq a_5 \leq a_3 \leq a_1$ and so the subsequence $\langle a_{2n-1} \rangle$ is monotonically decreasing and bounded below by $a_2$.

Thus $\langle a_{2n-1} \rangle$ is convergent. Let $\lim\limits_{n \to \infty} a_{2n-1} = l'$. ...(3)

We are given that $b_n = a_{2n-1} - a_{2n}$

$$\Rightarrow \lim_{n \to \infty} b_n = \lim_{n \to \infty} a_{2n-1} - \lim_{n \to \infty} a_{2n}$$

$$\Rightarrow \lim_{n \to \infty} b_n = l' - l, \text{ using (2) and (3)}$$

$$\Rightarrow 0 = l' - l, \text{ since } \lim_{n \to \infty} b_n = 0$$

$$\Rightarrow l = l'.$$

Thus the two subsequences $\langle a_{2n} \rangle$, $\langle a_{2n-1} \rangle$ of the given sequence $\langle a_n \rangle$ converge to the same limit $l$. Hence, by Example 96, the given sequence $\langle a_n \rangle$ also converges to $l$.

**Example 98.** *If $\langle S_n \rangle$ be a sequence of positive numbers such that $S_n = \frac{1}{2}(S_{n-1} + S_{n-2}) \ \forall \ n \geq 2$, then show that $\langle S_n \rangle$ converges. Also find $\lim\limits_{n \to \infty} S_n$.*

**Solution.** We have $\quad S_n = \frac{1}{2}(S_{n-1} + S_{n-2})$. ...(1)

**Let $S_1 = S_2$.**

By (1), $S_3 = \frac{1}{2}(S_2 + S_1) = S_1$ and so on.

In fact $S_n = S_1 \quad \forall \quad n$.

Thus the sequence $\langle S_n \rangle$ converges to $S_1$ in this case.

**Let $S_1 < S_2$.**

From (1), $\quad S_3 = \frac{1}{2}(S_2 + S_1) \Rightarrow S_3$ is A.M. of $S_1$ and $S_2$

$$\Rightarrow S_1 < S_3 < S_2. \qquad (\because \ S_1 < S_2).$$

Now $\quad S_4 = \frac{1}{2}(S_3 + S_2) \Rightarrow S_4$ is A.M. of $S_3$ and $S_2$

$$\Rightarrow S_3 < S_4 < S_2. \qquad (\because \ S_3 < S_2)$$

$S_5 = \frac{1}{2}(S_4 + S_3) \Rightarrow S_5$ is A.M. of $S_3$ and $S_4$

$$\Rightarrow S_3 < S_5 < S_4 \qquad (\because \ S_3 < S_4).$$

$$S_6 = \tfrac{1}{2}(S_5 + S_4) \quad \Rightarrow \quad S_5 < S_6 < S_4 \text{ etc.} \qquad (\because\ S_5 < S_4).$$

Thus $\qquad S_1 < S_3 < S_5 < \ldots < S_6 < S_4 < S_2.$

It follows that $\langle S_{2n-1} \rangle = \langle S_1, S_3, S_5, \ldots \rangle$ is monotonically increasing and bounded above by $S_2$, and $\langle S_{2n} \rangle = \langle S_2, S_4, S_6, \ldots \rangle$ is monotonically decreasing and bounded below by $S_1$.

Thus the two subsequences $\langle S_{2n-1} \rangle$ and $\langle S_{2n} \rangle$ are convergent. Suppose that

$$\lim_{n \to \infty} S_{2n-1} = l \quad \text{and} \quad \lim_{n \to \infty} S_{2n} = l'. \qquad \ldots(2)$$

From (1), $\qquad S_{2n} = \frac{1}{2}(S_{2n-1} + S_{2n-2}).$

Taking the limit as $n \to \infty$ and using (2), we obtain

$$l' = \frac{1}{2}(l + l') \quad \Rightarrow \quad l = l'.$$

Since both the subsequences $\langle S_{2n-1} \rangle$ and $\langle S_{2n} \rangle$ converge to $l$, so the given sequence $\langle S_n \rangle$ also converges to $l$. We have

$$\lim_{n \to \infty} S_n = l = \lim_{n \to \infty} S_{n-1}. \qquad \ldots(3)$$

Now we shall find the value of $l$.

We have $S_3 + \frac{1}{2} S_2 = \frac{1}{2}(S_2 + S_1) + \frac{1}{2} S_2 = \frac{1}{2}(S_1 + 2S_2),$

$$S_4 + \tfrac{1}{2} S_3 = \tfrac{1}{2}(S_3 + S_2) + \tfrac{1}{2} S_3 = S_3 + \tfrac{1}{2} S_2 = \tfrac{1}{2}(S_1 + 2S_2),$$

and so on.

In general, $\quad S_n + \frac{1}{2} S_{n-1} = \frac{1}{2}(S_1 + 2\,S_2).$

Taking the limit as $n \to \infty$ and using (3), we obtain

$$l + \tfrac{1}{2} l = \tfrac{1}{2}(S_1 + 2\,S_2) \quad \Rightarrow \quad l = \tfrac{1}{3}(S_1 + 2\,S_2).$$

Hence $\quad \lim_{n \to \infty} S_n = l = \frac{1}{3}(S_1 + 2\,S_2).$

Similarly, we can show that if $S_1 > S_2$, we get the same result.

**Example 99.** *Let $\langle a_n \rangle$ be a sequence of positive numbers : $a_0 = 0, a_1 = 1, a_n = \frac{1}{2}(a_{n-i} + a_{n-2})\ \forall\ n \geq 2$. Show that $\langle a_n \rangle$ converges and find its limit.* [D.U., B.Sc. (G) 1999]

**Solution.** If can be verified that

$$\langle a_{2n} \rangle = \langle 0, \tfrac{1}{2}, \tfrac{5}{8}, \tfrac{21}{32}, \ldots \rangle, \quad \langle a_{2n-1} \rangle = \langle 1, \tfrac{3}{4}, \tfrac{11}{16}, \ldots \rangle.$$

Clearly, $\langle a_{2n} \rangle$ is monotonically increasing and bounded above by 1, and $\langle a_{2n-1} \rangle$ is monotonically decreasing and bounded below by 0.

Thus the two subsequences are convergent.

Let $\lim a_{2n} = l$ and $\lim a_{2n-1} = l'$. From the given relation, we have

$$a_{2n} = \frac{1}{2}(a_{2n-1} + a_{2n-2})$$

$$\Rightarrow \quad \lim a_{2n} = \tfrac{1}{2}(\lim a_{2n-1} + \lim a_{2n-2})$$

$$\Rightarrow \quad l = \tfrac{1}{2}(l' + l) \Rightarrow l = l' \Rightarrow \lim a_n = l. \text{ Also } \lim a_{n-1} = l.$$

We have $a_2 + \frac{1}{2} a_1 = \frac{1}{2} + \frac{1}{2} = 1, a_3 + \frac{1}{2} a_2 = \frac{3}{4} + \frac{1}{4} = 1$ and so on.

In general, $a_n + \frac{1}{2} a_{n-1} = 1\ \forall\ n \in \mathbb{N}$. On taking the limit as $n \to \infty$,

$$l + \tfrac{1}{2} l = 1 \Rightarrow \tfrac{3}{2} l = 1 \Rightarrow l = \tfrac{2}{3}. \text{ Hence } \lim a_n = \tfrac{2}{3}.$$

**Example 100.** *If $\langle a_n \rangle$ be a sequence of positive numbers such that $a_n = \sqrt{a_{n-1}\, a_{n-2}}$ for $n > 2$ ; then show that the sequence converges to $(a_1\, a_2^2)^{1/3}$.* [D.U., Maths (H) 1991]

**Solution.** We have $\quad a_n = \sqrt{a_{n-1}\, a_{n-2}}, n > 2. \qquad \ldots(1)$

**Let $a_1 = a_2$.**

From (1), $a_3 = \sqrt{a_2 a_1} = \sqrt{a_1^2} = a_1$ and so on.

In fact, $a_n = a_1\ \forall\ n$ and so $\langle a_n \rangle$ converges to $a_1$.

**Let $a_1 < a_2$.**

From (1), $\quad a_3 = \sqrt{a_2\, a_1} \Rightarrow a_3$ is G.M. of $a_1$ and $a_2$

$\Rightarrow a_1 < a_3 < a_2$. $\qquad (\because\ a_1 < a_2)$

Now $\quad a_4 = \sqrt{a_3\, a_2} \Rightarrow a_4$ is G.M. of $a_3$ and $a_2$

$\Rightarrow a_3 < a_4 < a_2$, $\qquad (\because\ a_3 < a_2)$

$a_5 = \sqrt{a_4\, a_3} \Rightarrow a_3 < a_5 < a_4$, $\qquad (\because\ a_3 < a_4)$

$a_6 = \sqrt{a_5\, a_4} \Rightarrow a_5 < a_6 < a_4$ and so on. $\qquad (\because\ a_5 < a_4)$

Thus we obtain

$$a_1 < a_3 < a_5 < \ldots < a_6 < a_4 < a_2.$$

It follows that the subsequence $\langle a_{2n-1} \rangle = \langle a_1, a_3, a_5, \ldots \rangle$ is monotonically increasing and bounded above by $a_2$, and the subsequence $\langle a_{2n} \rangle = \langle a_2, a_4, a_6, \ldots \rangle$ is monotonically decreasing and bounded below by $a_1$.

Thus the two subsequences $\langle a_{2n-1} \rangle$ and $\langle a_{2n} \rangle$ are convergent. Let

$$\lim_{n \to \infty} a_{2n-1} = l' \quad \text{and} \quad \lim_{n \to \infty} a_{2n} = l. \qquad \ldots(2)$$

From (1), $\qquad a_{2n} = \sqrt{a_{2n-1}\, a_{2n-2}}.$

Squaring both sides and taking the limit as $n \to \infty$, we obtain

$$l^2 = l'\, l \Rightarrow l = l', \text{ using (2)}.$$

Since both the subsequences $\langle a_{2n-1} \rangle$ and $\langle a_{2n} \rangle$ converge to $l$, the given sequence $\langle a_n \rangle$ also converges to $l$. We have

*i.e.*, $\qquad \lim_{n \to \infty} a_n = l = \lim_{n \to \infty} a_{n-1}. \qquad \ldots(3)$

From (1) $a_n^2 a_{n-1} = a_{n-1}^2 a_{n-2}$.

Putting $n = n-1, n-2, \dots, 3$ in succession, we get

$$a_n^2 a_{n-1} = a_{n-1}^2 a_{n-2} = \dots = a_2^2 a_1$$

*i.e.*,

$$a_n^2 a_{n-1} = a_2^2 a_1.$$

Taking the limit and using (3), we obtain

$$l^3 = a_2^2 a_1 \Rightarrow l = (a_2^2 a_1)^{1/3}.$$

Hence $\lim\limits_{n\to\infty} a_n = (a_1 a_2^2)^{1/3}$. We obtain the same limit, if $a_1 > a_2$.

**Example 101.** *Let $\langle a_n \rangle$ be defined as follows :*

$$a_{n+2} = \sqrt{a_{n+2}\, a_n} \quad \forall\ n,\ a_1 > 0,\ a_2 > 0.$$

*Prove that $\langle a_n \rangle$ is convergent and obtain its limit.*

[D.U., Maths (H) 1998]

**Solution.** This is same as Example 100.

**Example 102.** *If $x_1, y_1$, are two positive unequal numbers and*

$$x_n = \tfrac{1}{2}(x_{n-1} + y_{n-1}) \quad \textit{and} \quad y_n = \sqrt{x_{n-1}\, y_{n-1}} \quad \forall \quad n \geq 2. \qquad \dots(1)$$

*Prove that the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ are monotonic and they converge to the same limit.*

**Solution.** Since $x_1$ and $y_1$ are unequal positive numbers, let $x_1 > y_1$

$$\Rightarrow \quad \tfrac{1}{2}(x_1 + y_1) > \sqrt{x_1 y_1} \qquad (\because \text{ A.M.} > \text{G.M.})$$

$$\Rightarrow \quad x_2 > y_2. \text{ Similarly, } \quad x_3 > y_3 \text{ and so on.}$$

Thus $\quad x_n > y_n \quad \forall \quad n. \qquad \dots(2)$

Also $\quad x_{n+1} = \tfrac{1}{2}(x_n + y_n) < \tfrac{1}{2}(x_n + x_n) = x_n$, by (2).

$\therefore \quad x_{n+1} < x_n \quad \forall \quad n. \qquad \dots(3)$

Thus $\langle x_n \rangle$ is a monotonically decreasing sequence.

Again $y_{n+1} = \sqrt{x_n y_n} > \sqrt{y_n y_n} = y_n$, by (2). $\therefore\ y_{n+1} > y_n\ \forall\ n. \qquad \dots(4)$

Thus $\langle y_n \rangle$ is a monotonically increasing sequence.

Now $x_n = \tfrac{1}{2}(x_{n-1} + y_{n-1}) > \tfrac{1}{2}(y_{n-1} + y_{n-1}) = y_{n-1} \Rightarrow x_n > y_{n-1}$, by (2)

$\Rightarrow x_n > y_{n-1} > y_{n-2} > \dots\dots > y_1$, using (4)

$\Rightarrow x_n > y_1\ \forall\ n \quad \Rightarrow y_1 < x_n\ \forall\ n$ and so $\langle x_n \rangle$ is bounded below.

Again, $\quad y_n = \sqrt{x_{n-1}\, y_{n-1}} < \sqrt{x_{n-1}\, x_{n-1}} = x_{n-1}$, using (2)

$\Rightarrow y_n < x_{n-1} < x_{n-2} < \dots\dots < x_1$, using (3)

$\Rightarrow y_n < x_1 \quad \forall \quad n$ and so $\langle y_n \rangle$ is bounded above.

Since $\langle x_n \rangle$ is monotonically decreasing and bounded below, therefore, $\langle x_n \rangle$ is convergent.

Let $\qquad \lim\limits_{n\to\infty} x_n = l. \qquad \dots(5)$

Since $\langle y_n \rangle$ is monotonically increasing and bounded above, therefore, $\langle y_n \rangle$ is convergent.

Let $\qquad \lim\limits_{n\to\infty} y_n = l'. \qquad \dots(6)$

Now we shall show that $l = l'$.

We have $\quad x_n = \dfrac{1}{2}(x_{n-1} + y_{n-1}) \quad \Rightarrow \quad 2x_n = x_{n-1} + y_{n-1}$

$$\Rightarrow \quad 2 \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-1} + \lim_{n\to\infty} y_{n-1}$$

$$\Rightarrow \quad 2l = l + l' \Rightarrow l = l', \text{ by (5) and (6).}$$

Hence the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ are monotonic and converge to the same limit.

**Example 103.** *If $a_1 > 0$, $a_2 > 0$ and*

$$a_n = \frac{2a_{n-1}\, a_{n-2}}{a_{n-1} + a_{n-2}}, \quad n > 2\ ; \qquad \dots(1)$$

*then show that $\langle a_n \rangle$ converges to $\dfrac{3a_1 a_2}{2a_1 + a_2}$.*

**Solution.** If $a_1 = a_2$, then by (1), $a_n = a_1\ \forall\ n$.

Thus $\langle a_n \rangle$ converges to $a_1$.

**Let $a_1 < a_2$.**

From (1), $\quad a_3 = \dfrac{2\, a_2\, a_1}{a_2 + a_1} \quad \Rightarrow \quad a_3$ is H.M. of $a_1$ and $a_2$

$$\Rightarrow \quad a_1 < a_3 < a_2. \qquad (\because\ a_1 < a_2)$$

Now $\quad a_4 = \dfrac{2\, a_3\, a_2}{a_3 + a_2} \quad \Rightarrow \quad a_4$ is H.M. of $a_3$ and $a_2$.

$$\Rightarrow \quad a_3 < a_4 < a_2, \qquad (\because\ a_3 < a_2).$$

$$a_5 = \frac{2\, a_4\, a_3}{a_4 + a_3} \quad \Rightarrow \quad a_3 < a_5 < a_4 \text{ and so on.}$$

Thus $\quad a_1 < a_3 < a_5 < \dots < a_6 < a_4 < a_2$.

It follows that $\langle a_{2n-1} \rangle$ is monotonically increasing and bounded above by $a_2$ and $\langle a_{2n} \rangle$ is monotonically decreasing and bounded below by $a_1$.

Thus $\langle a_{2n-1} \rangle$ and $\langle a_{2n} \rangle$ are convergent sequences.

Let $\quad \lim\limits_{n\to\infty} a_{2n-1} = l \quad$ and $\quad \lim\limits_{n\to\infty} a_{2n} = l'. \qquad \dots(2)$

From (1), $\quad a_{2n} = \dfrac{2\, a_{2n-1}\, a_{2n-2}}{a_{2n-1} + a_{2n-2}}$.

Taking the limit as $n \to \infty$ and using (2), we obtain

$$l' = \frac{2ll'}{l + l'} \Rightarrow l = l'.$$

Since both the subsequences $\langle a_{2n-1} \rangle$ and $\langle a_{2n} \rangle$ converge to $l$, the given sequence $\langle a_n \rangle$ also converges to $l$. We have

$$\lim_{n \to \infty} a_n = l = \lim_{n \to \infty} a_{n-1}. \qquad \ldots(3)$$

Now we shall find the value of $l$. We have

$$\frac{1}{a_2} + \frac{1}{2a_1} = \frac{2a_1 + a_2}{2a_1 a_2},$$

$$\frac{1}{a_3} + \frac{1}{2a_2} = \frac{a_2 + a_1}{2a_2 a_1} + \frac{1}{2a_2} = \frac{2a_1 + a_2}{2a_1 a_2},$$

$$\frac{1}{a_4} + \frac{1}{2a_3} = \frac{a_3 + a_2}{2a_3 a_2} + \frac{1}{2a_3} = \frac{a_3 + 2a_2}{2a_2 a_3} = \frac{1}{a_3} + \frac{1}{2a_2} = \frac{2a_1 + a_2}{2a_1 a_2}$$

and so on.

In general, $$\frac{1}{a_n} + \frac{1}{2a_{n-1}} = \frac{2a_1 + a_2}{2a_1 a_2} \quad \forall \; n$$

or $$\frac{2a_{n-1} + a_n}{2a_n a_{n-1}} = \frac{2a_1 + a_2}{2a_1 a_2}.$$

Taking the limit as $n \to \infty$ and using (3), we get

$$\frac{2l + l}{2l^2} = \frac{2a_1 + a_2}{2a_1 a_2} \quad \text{or} \quad \frac{3}{2l} = \frac{2a_1 + a_2}{2a_1 a_2} \quad \text{or} \quad l = \frac{3a_1 a_2}{2a_1 + a_2}.$$

Hence $$\lim_{n \to \infty} a_n = \frac{3a_1 a_2}{2a_1 + a_2}.$$

## EXERCISES

1. If $0 < u_1 < u_2$ and $\dfrac{2}{u_{n+2}} = \dfrac{1}{u_{n+1}} + \dfrac{1}{u_n}$ for $n \geq 1$, prove that $\langle u_n \rangle$ is convergent.

   [**Hint.** From the given relation, we have

   $$\frac{2}{u_{n+2}} = \frac{u_n + u_{n+1}}{u_n u_{n+1}} \quad \text{or} \quad u_{n+2} = \frac{2 u_n u_{n+1}}{u_n + u_{n+1}} \quad \text{for} \quad n \geq 1$$

   Now it is same as Example 103.]

2. If $0 < u_1 < u_2$ and $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for $n \geq 1$, prove that $\langle u_n \rangle$ is convergent.

   [**Hint.** Similar to Example 99]

3. If $0 < u_1 < u_2$ and $u_{n+2} = \sqrt{u_{n+1} u_n}$ for $n \geq 1$, prove that $\langle u_n \rangle$ is convergent.

   [**Hint.** Similar to Example 100]

4. Prove that the sequence $\langle u_n \rangle$ is convergent by showing that the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ converge to the same limit :

   $$u_1 = 1 \text{ and } u_{n+1} = \frac{6}{1 + u_n} \text{ for } n \geq 1.$$

5. Prove that the sequence $\langle u_n \rangle$ defined by $0 < u_1 < u_2$ and

   $$u_{n+2} = \frac{2 u_{n+1} + u_n}{3} \text{ for } n \geq 1 \text{ converges to } \frac{u_1 + 3u_2}{4}.$$

6. Prove that the sequence $\langle u_n \rangle$ defined by $0 < u_1 < u_2$ and

   $$u_{n+2} = \frac{u_{n+1} + 2 u_n}{3} \text{ for } n \geq 1 \text{ converges to } \frac{2u_1 + 3u_2}{5}.$$

### 3.12 Limit Superior and Limit Inferior

Recall that Bolzano-Weierstrass Theorem states :

*Every bounded sequence must have a limit point.*

**Definition 1.** *The greatest limit point of a bounded sequence $\langle a_n \rangle$ is called the* **limit superior** *of $\langle a_n \rangle$. It is written as*

$$\lim_{n \to \infty} \sup a_n \quad \text{or} \quad \overline{\lim_{n \to \infty}} \, a_n.$$

**Definition 2.** *The smallest limit point of a bounded sequence $\langle a_n \rangle$ is called the* **limit inferior** *of $\langle a_n \rangle$. It is written as*

$$\lim_{n \to \infty} \inf a_n \quad \text{or} \quad \underline{\lim_{n \to \infty}} \, a_n.$$

**Remark.** *It is clear that* $\displaystyle\lim_{n \to \infty} \inf a_n \leq \lim_{n \to \infty} \sup a_n.$

**Illustrations**

1. Let $\langle a_n \rangle = \langle (-1)^n \rangle = \langle -1, 1, -1, 1, \ldots \rangle$.

   Its set of limit points $= \{-1, 1\}$.

   The greatest and the smallest limit points are 1 and $-1$, respectively.

   $\therefore \quad \lim \sup a_n = 1, \lim \inf a_n = -1$.

2. Let $\langle a_n \rangle = \langle 1 + (-1)^n \rangle = \langle 0, 2, 0, 2, \ldots \rangle$.

   Its set of limit points $= \{0, 2\}$.

   $\therefore \quad \lim \sup a_n = 2, \lim \inf a_n = 0$.

3. Let $\langle a_n \rangle = \langle \frac{1}{n} \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$.

   It has exactly one limit point *viz.* 0.

   $\therefore \quad \lim \sup a_n = 0, \lim \inf a_n = 0$.

**Definition 3.** *If a sequence $\langle a_n \rangle$ is not bounded above, then we define $\displaystyle\lim_{n \to \infty} \sup a_n = +\infty$ and we say that $+\infty$ is the greatest limit point.*

**Definition 4.** *If a sequence $\langle a_n \rangle$ is not bounded below, then we define $\displaystyle\lim_{n \to \infty} \inf a_n = -\infty$ and we say that $-\infty$ is the smallest limit point.*

**Definition 5.** *If a sequence $\langle a_n \rangle$ is bounded below but not bounded above and has no other limit point except $+\infty$, then we define*

$$\lim_{n \to \infty} \inf a_n = \lim_{n \to \infty} \sup a_n = +\infty.$$

**Definition 6.** *If a sequence $\langle a_n \rangle$ is bounded above but not bounded below and has no other limit point except $-\infty$, then we define*

$$\lim_{n \to \infty} \inf a_n = \lim_{n \to \infty} \sup a_n = -\infty.$$

**Illustrations**

**4.** Let $\langle a_n \rangle = \langle n^2 \rangle = \langle 1, 4, 9, 16, \ldots \rangle$.

Then $\langle a_n \rangle$ is bounded below but not bounded above.

$\therefore \quad \lim\limits_{n \to \infty} \inf a_n = \lim\limits_{n \to \infty} \sup a_n = +\infty$. (Definition 5)

(Note that $\langle a_n \rangle$ has no other limit point except $+\infty$)

**5.** Let $\langle a_n \rangle = \langle -n^2 \rangle = \langle -1, -4, -9, -16, \ldots \rangle$.

Then $\langle a_n \rangle$ is bounded above but not bounded below.

$\therefore \quad \lim\limits_{n \to \infty} \inf a_n = \lim\limits_{n \to \infty} \sup a_n = -\infty$. (Definition 6)

**6.** Let $\langle a_n \rangle = \langle 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots \rangle$.

Then $\langle a_n \rangle$ is bounded below but not bounded above.

Further 0 is the limit point of $\langle a_n \rangle$.

$\therefore \quad \lim\limits_{n \to \infty} \inf a_n = 0, \ \lim\limits_{n \to \infty} \sup a_n = +\infty$ (Definition 3).

**7.** Let $\langle a_n \rangle = \langle -1, -2, -\frac{1}{2}, -3, -\frac{1}{3}, \ldots \rangle$.

Then $\lim \inf a_n = -\infty$ and $\lim \sup a_n = 0$. (Definition 4)

**Theorem 22.** *For a bounded sequence $\langle a_n \rangle$, show that $\lim\limits_{n \to \infty} a_n = l$ if and only if for each $\varepsilon > 0$*

*(i) $a_n > l - \varepsilon$ for infinitely many values of $n$,*

*(ii) $a_n < l + \varepsilon$ for all except finitely many values of $n$.*

[D.U., B.Sc. (G) 1996 ; Maths. (H) 1989]

**Proof.** *The condition is necessary.*

Let $\lim\limits_{n \to \infty} \sup a_n = l$, so that $l$ is the greatest limit point of $\langle a_n \rangle$. We shall prove conditions (*i*) and (*ii*).

Since $l$ is a limit point of $\langle a_n \rangle$, for any $\varepsilon > 0$,

$$a_n \in \,]\, l - \varepsilon, l + \varepsilon \,[\, \text{ for infinitely many values of } n$$

$$\Rightarrow \quad a_n > l - \varepsilon \quad \text{for infinitely many values of } n.$$

This proves condition (*i*).

Since $l$ is the greatest limit point of $\langle a_n \rangle$ and $l + \varepsilon > l$, so $l + \varepsilon$ cannot be a limit point of $\langle a_n \rangle$. Then

$$a_n \geq l + \varepsilon \text{ for only finitely many values of } n, \qquad \ldots(1)$$

for if $a_n \geq l + \varepsilon$ for infinitely many values of $n$, then $\langle a_n \rangle$ will have a limit point say $p \geq l + \varepsilon$, which is impossible, since $l$ is the greatest limit point of $\langle a_n \rangle$.

From (1), it follows that

$$a_n < l + \varepsilon \text{ for all except finitely many values of } n.$$

This proves condition (*ii*).

*The condition is sufficient.*

Suppose conditions (*i*) and (*ii*) are given.

We shall show that $l = \lim\limits_{n \to \infty} \sup a_n$.

From conditions (*i*) and (*ii*), $l - \varepsilon < a_n < l + \varepsilon$ for infinitely many values n and $\forall \ \varepsilon > 0$

$\Rightarrow \quad a_n \in \,]\, l - \varepsilon, l + \varepsilon \,[\,$ for infinitely many values of $n$

$\Rightarrow \quad l$ is a limit point of $\langle a_n \rangle$.

Now we show that $l$ is the greatest limit point of $\langle a_n \rangle$.

Suppose $l' > l$.

Let $p$ and $q$ be two numbers such that

$$l < p < l' < q. \qquad \ldots(2)$$

Let $\varepsilon = p - i > 0$. Then $l + \varepsilon = p$.

Using (*ii*), $a_n < l + \varepsilon = p$ for all except finitely many values of $n$

$\Rightarrow \quad a_n \geq p$ for only finitely many values of $n$.

From (2), we see that $]\, p, q \,[$ is a *nbd.* of $l'$ such that

$$p \leq a_n \text{ for only finitely many values of } n$$

$\Rightarrow l'$ is not a limit point of $\langle a_n \rangle$, where $l' > l$

$\Rightarrow l$ is the greatest limit point of $\langle a_n \rangle$.

Hence $\lim\limits_{n \to \infty} \sup a_n = l$.

**Theorem 23.** *For a bounded sequence $\langle a_n \rangle$, show that $\lim\limits_{n \to \infty} \inf a_n = l$ if and only if for each $\varepsilon > 0$*

*(i) $a_n < l + \varepsilon$ for infinitely many values of $n$,*

*(ii) $a_n > l - \varepsilon$ for all except finitely many values of $n$.*

[D.U., B.Sc. (G) 1995 ; Maths (H) 1992, 90]

**Proof.** *The condition is necessary.*

Let $\lim\limits_{n \to \infty} \inf a_n = l$, so that $l$ is the smallest limit point of $\langle a_n \rangle$. We shall ove conditions (*i*) and (*ii*).

Let $\varepsilon > 0$ be given. Since $l$ is a limit point of $\langle a_n \rangle$, therefore

$$a_n \in \,]\, l - \varepsilon, l + \varepsilon \,[\, \text{ for infinitely many values of } n$$

$\Rightarrow a_n < l + \varepsilon$ for infinitely many values of $n$.

This proves condition (*i*).

Since $l$ is the smallest limit point of $\langle a_n \rangle$ and $l - \varepsilon < l$, so $l - \varepsilon$ cannot a limit point of $\langle a_n \rangle$. Thus

$$a_n \leq l - \varepsilon \text{ for only finitely many values of } n, \qquad \ldots(1)$$

for if $a_n \leq l - \varepsilon$ for infinitely many values of $n$, then $\langle a_n \rangle$ will have a limit point say $p \leq l - \varepsilon$, which is impossible, since $l$ is the smallest limit point of $\langle a_n \rangle$. From (1), it follows that

$$a_n > l - \varepsilon \text{ for all except finitely many values of } n.$$

This proves condition (*ii*).

*The condition is sufficient.*

Suppose we are given the two conditions (*i*) and (*ii*).

We shall show that $\lim\limits_{n \to \infty} \inf a_n = l$.

From (*i*) and (*ii*), $l - \varepsilon < a_n < l + \varepsilon$, for infinitely many values of $n$ and for each $\varepsilon > 0$

$\Rightarrow a_n \in \, ]\, l - \varepsilon, l + \varepsilon \,[$ for infinitely many values of $n$

$\Rightarrow l$ is a limit point of $\langle a_n \rangle$.

Now we show that $l$ is the smallest limit point of $\langle a_n \rangle$.

Suppose $l' < l$.

Let $p$ and $q$ be two numbers such that

$$p < l' < q < l. \qquad \text{...(2)}$$

Let $\varepsilon = l - q > 0$. Then $l - \varepsilon = q$.

Using (*ii*), $a_n > l - \varepsilon = q$, for all except finitely many values of $n$

$\Rightarrow \quad a_n \leq q$ for only finitely many values of $n$.

From (2), we see that $]\, p, q \,[$ is a *nbd.* of $l'$ such that

$a_n \leq q$ for only finitely many values of $n$

$\Rightarrow l'$ is *not* a limit point of $\langle a_n \rangle$, where $l' < l$

$\Rightarrow l$ is the smallest limit point of $\langle a_n \rangle$.

Hence $\lim\limits_{n \to \infty} \inf a_n = l$.

**Theorem 24.** *A bounded sequence $\langle a_n \rangle$ converges to $l$ iff*

$$\lim_{n \to \infty} \sup a_n = \lim_{n \to \infty} \inf a_n = l.$$

[D.U., Maths (H) 1998, 1996, 92 ; B.Sc. (G) 1997]

**Proof.** *The condition is necessary.*

Suppose $\lim\limits_{n \to \infty} a_n = l$.

Then $l$ is the unique limit point of $\langle a_n \rangle$ [Theorem 11]

*i.e.*, $\quad l$ is the greatest as well as the smallest limit point of $\langle a_n \rangle$.

Hence $\lim\limits_{n \to \infty} \sup a_n = \lim\limits_{n \to \infty} \inf a_n = l$.

*The condition is sufficient.*

Suppose $\lim\limits_{n \to \infty} \sup a_n = \lim\limits_{n \to \infty} \inf a_n = l$.

We shall show $\lim\limits_{n \to \infty} a_n = l$. Let $\varepsilon > 0$ be given.

Since $\lim\limits_{n \to \infty} \sup a_n = l$, therefore, by Theorem 22, we have

$$a_n < l + \varepsilon \text{ for all except finitely many values of } n.$$

Thus there exists a positive integer $m_1$ such that

$$a_n < l + \varepsilon \quad \forall \quad n \geq m_1. \qquad \text{...(1)}$$

Since $\lim\limits_{n \to \infty} \inf a_n = l$, therefore, by Theorem 23, we have

$$l - \varepsilon < a_n \text{ for all except finitely many values of } n.$$

Thus there exists a positive integer $m_2$ such that

$$l - \varepsilon < a_n \quad \forall \quad n \geq m_2. \qquad \text{...(2)}$$

Let $m = \max \{m_1, m_2\}$. From (1), (2) ; we get

$$l - \varepsilon < a_n < l + \varepsilon \quad \forall \quad n \geq m$$

or $\quad |a_n - l| < \varepsilon \quad \forall \quad n \geq m$.

Hence $\lim\limits_{n \to \infty} a_n = l$.

**Remark.** *For a convergent sequence $\langle a_n \rangle$, we observe that*

$$\lim_{n \to \infty} \sup a_n = \lim_{n \to \infty} \inf a_n = \lim_{n \to \infty} a_n .$$

**Ex.** *Show that if for a bounded sequence the limit inferior and the limit superior are equal, then the sequence is convergent.*

[D.U., B.Sc. (G) 1986]

[**Hint.** Prove the *sufficient condition* of the above theorem].

## EXAMPLES

**Example 104.** *Find the limit superior and the limit inferior of each of the following sequences :*

[D.U., Maths (H) 1998]

(*a*) $\langle (-1)^n \, (1 + \frac{1}{n}) \rangle$

$$= \langle -2, 1\tfrac{1}{2}, -1\tfrac{1}{3}, 1\tfrac{1}{4}, -1\tfrac{1}{5}, 1\tfrac{1}{6}, \ldots \rangle.$$

Its set of limit points is $\{-1, 1\}$. Consequently,

limit superior = greatest limit point = 1,

limit inferior = smallest limit point = − 1.

(*b*) $\quad \langle n \, (1 + (-1)^n) \rangle = \langle 0, 4, 0, 8, 0, 12, \ldots \rangle$.

This sequence is bounded below but not bounded above.

Its limit point is 0.

$\therefore \quad$ limit superior $= + \infty$, limit inferior $= 0$.

(*c*) Let $a_n = \frac{(-1)^n}{n^2}$. Then $\lim_{n \to \infty} a_n = 0$.

Hence $\lim_{n \to \infty} \sup a_n = \lim_{n \to \infty} \inf a_n = 0$.

**Example 106.** *If* $\lim_{n \to \infty} \sup a_n = M$, *then prove that*

(*a*) *there is a subsequence of* $\langle a_n \rangle$ *that converges to* $M$,

(*b*) *the limit superior of no subsequence of* $\langle a_n \rangle$ *can exceed* $M$.

(*c*) *no subsequence of* $\langle a_n \rangle$ *can converge to a number greater than* $M$.

**Solution.** (*a*) Since $\lim_{n \to \infty} \sup a_n = M$, $M$ is the greatest limit point of $\langle a_n \rangle$.

Since $M$ is a limit point of $\langle a_n \rangle$, so there exists a subsequence $\langle a_{n_k} \rangle$ of $\langle a_n \rangle$ such that $a_{n_k} \to M$. (Theorem 21)

(*b*) Suppose $\langle a_{n_k} \rangle$ is a subsequence of $\langle a_n \rangle$ such that

$$\lim \sup a_{n_k} = M_1, \quad \text{where} \quad M_1 > M.$$

Then $M_1$ is the greatest limit point of $\langle a_{n_k} \rangle$. Consequently, $M_1$ is a limit point of $\langle a_n \rangle$, where $M_1 > M$. This contradicts the fact that $M$ is the greatest limit point of $\langle a_n \rangle$.

Hence $\lim \sup a_{n_k} = M_1$, $M_1 \leq M$.

(*c*) Let $\langle a_{n_k} \rangle$ be a subsequence of $\langle a_n \rangle$ such that

$$a_{n_k} \to M_2, \quad \text{where} \quad M_2 > M$$

Then $M_2$ must be a limit point of $\langle a_n \rangle$ (Theorem 21), where $M_2 > M$. This contradicts the fact that $M$ is the greatest limit point of $\langle a_n \rangle$.

Hence $a_{n_k} \to M_2$, where $M_2 \leq M$.

**Example 107.** *If* $\lim_{n \to \infty} \inf a_n = m$, *then prove that*

(*a*) *there is a subsequence of* $\langle a_n \rangle$ *that converges to* $m$,

(*b*) *the limit inferior of no subsequence of* $\langle a_n \rangle$ *can be less than* $m$,

(*c*) *no subsequence of* $\langle a_n \rangle$ *can converge to a number less than* $m$.

[D.U., Maths (H) 1999]

**Solution.** By definition, $m$ is the smallest limit point of $\langle a_n \rangle$. Now proceed like Example 106.

**Example 108.** *If* $\langle a_n \rangle$ *and* $\langle b_n \rangle$ *are bounded sequences such that* $a_n \leq b_n \ \forall \ n \in \mathbf{N}$, *then prove that*

(*i*) $\lim_{n \to \infty} \sup a_n \leq \lim_{n \to \infty} \sup b_n$. [D.U., Maths (H) 1995]

(*ii*) $\lim_{n \to \infty} \inf a_n \leq \lim_{n \to \infty} \inf b_n$.

**Solution.** (*i*) Let $\lim_{n \to \infty} \sup b_n = b$.

By Theorem 22, for any $\varepsilon > 0$, $\exists$ a positive integer $m$ such that

$$b_n < b + \varepsilon \quad \forall \quad n \geq m$$

$$\Rightarrow \qquad a_n < b + \varepsilon \quad \forall \quad n \geq m \qquad (\because\ a_n \leq b_n \ \forall\ n)$$

$\Rightarrow$ all limit points of $\langle a_n \rangle$ are $\leq b + \varepsilon$

$\Rightarrow$ the greatest limit point of $\langle a_n \rangle$ is $\leq b + \varepsilon$

$\Rightarrow \lim_{n \to \infty} \sup a_n \leq \lim_{n \to \infty} \sup b_n + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $\lim_{n \to \infty} \sup a_n \leq \lim_{n \to \infty} \sup b_n$.

(*i*) Let $\lim_{n \to \infty} \inf a_n = a$.

By Theorem 23, for any $\varepsilon > 0$, $\exists$ a positive integer $m$ such that

$$a_n > a - \varepsilon \quad \forall \quad n \geq m$$

$$\Rightarrow \qquad b_n > a - \varepsilon \quad \forall \quad n \geq m \qquad (\because\ b_n \geq a_n \ \forall\ n)$$

$\Rightarrow$ all limit points of $\langle b_n \rangle$ are $\geq a - \varepsilon$

$\Rightarrow$ the smallest limit point of $\langle b_n \rangle$ is $\geq a - \varepsilon$

$\Rightarrow \lim_{n \to \infty} \inf b_n \geq \lim_{n \to \infty} \inf a_n - \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $\lim_{n \to \infty} \inf b_n \geq \lim_{n \to \infty} \inf a_n$.

Hence $\lim_{n \to \infty} \inf a_n \leq \lim_{n \to \infty} \inf b_n$.

**Example 109.** *If $\langle a_n \rangle$ and $\langle b_n \rangle$ are bounded sequences, then*

(*i*) $\lim_{n \to \infty} \sup (a_n + b_n) \leq \lim_{n \to \infty} \sup a_n + \lim_{n \to \infty} \sup b_n$.

[D.U., B.Sc. (G) 1998 ; Maths (H) 1997]

(*ii*) $\lim_{n \to \infty} \inf (a_n + b_n) \geq \lim_{n \to \infty} \inf a_n + \lim_{n \to \infty} \inf b_n$.

**Solution.** (*i*) Let $\lim_{n \to \infty} \sup a_n = a$ and $\lim_{n \to \infty} \sup b_n = b$.

By Theorem 22, for any $\varepsilon > 0$, there exist positive integers $m_1$ and $m_2$ such that

$$a_n < a + \varepsilon/2 \ \forall\ n \geq m_1, \ b_n < b + \varepsilon/2 \ \forall\ n \geq m_2.$$

Let $m = \max(m_1, m_2)$ so that $m \geq m_1$ and $m \geq m_2$.

Then $a_n < a + \varepsilon/2$ and $b_n < b + \varepsilon/2 \quad \forall \quad n \geq m$

$\Rightarrow a_n + b_n < a + b + \varepsilon \quad \forall \quad n \geq m$

$\Rightarrow a_n + b_n < a + b + \varepsilon \quad \forall \quad n \geq m$

$\Rightarrow$ all limit points of $\langle a_n + b_n \rangle$ are $\leq a + b + \varepsilon$

$\Rightarrow$ the greatest limit point of $\langle a_n + b_n \rangle$ is $\leq a + b + \varepsilon$

$\Rightarrow \lim_{n \to \infty} \sup (a_n + b_n) \leq \lim_{n \to \infty} \sup a_n + \lim_{n \to \infty} \sup b_n + \varepsilon$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{n \to \infty} \sup (a_n + b_n) \leq \lim_{n \to \infty} \sup a_n + \lim_{n \to \infty} \sup b_n.$$

(*ii*) Let $\lim_{n \to \infty} \inf a_n = a'$ and $\lim_{n \to \infty} \inf b_n = b'$.

By Theorem 23, for any $\varepsilon > 0$, there exist positive integers $m_1$ and $m_2$ such that

$$a_n > a' - \varepsilon/2 \quad \forall \quad n \geq m_1 \text{ and } b_n > b' - \varepsilon/2 \quad \forall \quad n \geq m_2.$$

Let $m = \max(m_1, m_2)$. Then

$$a_n > a' - \varepsilon/2 \text{ and } b_n > b' - \varepsilon/2 \ \forall\ n \geq m$$

$$\Rightarrow \qquad a_n + b_n > a' + b' - \varepsilon \quad \forall \quad n \geq m$$

$\Rightarrow$ all limit points of $\langle a_n + b_n \rangle$ are $\geq a' + b' - \varepsilon$

$\Rightarrow$ the smallest limit point of $\langle a_n + b_n \rangle$ is $\geq a' + b' - \varepsilon$

$\Rightarrow \lim_{n \to \infty} \inf (a_n + b_n) \geq \lim_{n \to \infty} \inf a_n + \lim_{n \to \infty} b_n - \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{n \to \infty} \inf (a_n + b_n) \geq \lim_{n \to \infty} \inf a_n + \lim_{n \to \infty} \inf b_n.$$

**Remark.** *Strict inequality may hold in* (*i*) *and* (*ii*) *of Example 109* as seen by the following example :

Let $\langle a_n \rangle = \langle -1, 0, 1, -1, 0, 1, \ldots \rangle$, $\langle b_n \rangle = \langle 0, 1, -1, 0, 1, -1, \ldots \rangle$,

Then $\langle a_n + b_n \rangle = \langle -1, 1, 0, -1, 1, 0, \ldots \rangle$. We see that

$$\lim \sup a_n = 1, \ \lim \sup b_n = 1, \ \lim \sup (a_n + b_n) = 1,$$

$$\lim \inf a_n = -1, \ \lim \inf b_n = -1, \ \lim \inf (a_n + b_n) = -1.$$

It is clear that

$$\lim \sup (a_n + b_n) < \lim \sup a_n + \lim \sup b_n,$$

$$\lim \inf (a_n + b_n) > \lim \inf a_n + \lim \inf b_n.$$

**Example 110.** *If $\langle a_n \rangle$ is any bounded sequence, then*

(*a*) *$\lim \inf (-a_n) = -\lim \sup a_n$ or $\underline{\lim}\,(-a_n) = -\overline{\lim}\, a_n$,*

(*b*) *$\lim \sup (-a_n) = -\lim \inf a_n$ or $\overline{\lim}\,(-a_n) = -\underline{\lim}\, a_n$.*

**Solution.** Let $\lim \sup a_n = a$. By Theorem 22, for any $\varepsilon > 0$,

(*i*) $a_n > a - \varepsilon$ for infinitely many values of $n$,

(*ii*) $a_n < a + \varepsilon$ for all except finitely many values of $n$.

Thus $\quad -a_n < -a + \varepsilon$ for infinitely many values of $n$,

and $\quad -a_n > -a - \varepsilon$ for all except finitely many values of $n$.

By virtue of Theorem 23, it follows that $\lim \inf (-a_n) = -a$.

Hence $\quad \lim \inf (-a_n) = -\lim \sup a_n$.

Similarly, we can prove the second part.

**Solution.** By Example 109(*i*), we know

$$\overline{\lim}\,(a_n+b_n) \le \overline{\lim}\,a_n + \overline{\lim}\,b_n\,.$$

Since $\langle b_n \rangle$ is convergent, $\overline{\lim}\,b_n = \lim b_n\,.$ ( See Theorem 24)

$$\therefore \qquad \overline{\lim}\,(a_n+b_n) \le \overline{\lim}\,a_n + \lim b_n\,. \qquad \text{...(1)}$$

Now we show $\overline{\lim}\,a_n + \lim b_n \le \overline{\lim}\,(a_n+b_n)$.

Let $\lim b_n = b$ and $\overline{\lim}\,a_n = a$, so that $a$ is the greatest limit point of $\langle a_n \rangle$. Since $a$ is a limit point of $\langle a_n \rangle$, there exists a subsequence $\langle a_{n_k} \rangle$ of $\langle a_n \rangle$ such that $a_{n_k} \to a$. Since $\langle b_n \rangle$ is convergent to $b$, $b$ is the unique limit point of $\langle b_n \rangle$. Consequently, there exists a subsequence $\langle b_{n_k} \rangle$ of $\langle b_n \rangle$ which converges to $b$. Thus $a_{n_k} + b_{n_k} \to a+b$

$\Rightarrow a+b$ is a limit point of $\langle a_n + b_n \rangle$

$\Rightarrow a+b \le \overline{\lim}\,(a_n+b_n)$, the greatest limit point of $\langle a_n + b_n \rangle$

$$\Rightarrow \overline{\lim}\,a_n + \lim b_n \le \overline{\lim}\,(a_n+b_n). \qquad \text{...(2)}$$

Hence $\overline{\lim}\,(a_n+b_n) = \overline{\lim}\,a_n + \lim b_n$, by (1) and (2).

**Example 113.** *Let $\langle a_n \rangle$ be a real bounded sequence and $\langle b_n \rangle$ a real convergent sequence. Show that*

$$\liminf\,(a_n+b_n) = \liminf\,a_n + \lim b_n.$$

*Or*

$$\underline{\lim}\,(a_n+b_n) = \underline{\lim}\,a_n + \lim b_n.$$

**Solution.** By Example 109(*ii*), we know

$$\underline{\lim}\,(a_n+b_n) \ge \underline{\lim}\,a_n + \underline{\lim}\,b_n = \underline{\lim}\,a_n + \lim b_n, \qquad \text{...(1)}$$

since $\langle b_n \rangle$ is convergent $\Rightarrow \underline{\lim}\,b_n = \lim b_n$. (See Theorem 24)

Let $\lim b_n = b$ and $\underline{\lim}\,a_n = a$, so that $a$ is the smallest limit point of $\langle a_n \rangle$. There exists a subsequence $\langle a_{n_k} \rangle$ of $\langle a_n \rangle$ such that $a_{n_k} \to a$.

Since $\langle b_n \rangle$ is convergent to $b$, $b$ is the unique limit point of $\langle b_n \rangle$. Consequently, there exists a subsequence $\langle b_{n_k} \rangle$ of $\langle b_n \rangle$ which converges to $b$ *i.e.*, $b_{n_k} \to b$. Also $a_{n_k} \to a$.

$$\therefore \qquad a_{n_k} + b_{n_k} \to a+b$$

$\Rightarrow a+b$ is a limit point of $\langle a_n + b_n \rangle$

$\Rightarrow a+b \ge \underline{\lim}\,(a_n+b_n)$, since $\underline{\lim}\,(a_n+b_n)$ is the smallest limit point of $\langle a_n + b_n \rangle$

$$\Rightarrow \underline{\lim}\,a_n + \lim b_n \ge \underline{\lim}\,(a_n+b_n). \qquad \text{...(2)}$$

From (1) and (2), $\underline{\lim}\,(a_n+b_n) = \underline{\lim}\,a_n + \lim b_n$.

# 4

# Infinite Series

## 4.1 Infinite Series

*An expression of the form*

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots,$$

*where each $a_n$ is a real number, is called an* ***infinite series*** *of real numbers and is denoted by* $\sum\limits_{n=1}^{\infty} a_n$ *or* $\Sigma\, a_n$ . $a_n$ *is called the nth term of the series* $\Sigma\, a_n$ .

**Illustrations**

**1.** $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$

**2.** $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n\,!} = x + \dfrac{x^2}{2\,!} + \dfrac{x^3}{3\,!} + \ldots$

**3.** $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \ldots$

## 4.2 Convergence and Divergence of an Infinite Series

Suppose $\Sigma\, a_n$ is an infinite series.

We define a sequence $\langle S_n \rangle$ as follows :

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots \qquad \vdots$$
$$S_n = a_1 + a_2 + \ldots + a_n, \text{ and so on.}$$

*The sequence $\langle S_n \rangle$ is called the* **sequence of partial sums** *of the series* $\Sigma\, a_n$ .

**Definitions.** *A series $\Sigma\, a_n$ is said to be* **convergent**, *if the sequence $\langle S_n \rangle$ of partial sums of $\Sigma\, a_n$ is convergent. If* $\lim\limits_{n \to \infty} S_n = S$, *then S is called the* **sum of the series** $\Sigma\, a_n$ . *We write* $S = \sum\limits_{n=1}^{\infty} a_n$ .

G.S:- Sum of n terms of a G.P with first term 'a' and common ratio 'r' is given by $S_n = a\left(\frac{1-x^n}{1-x}\right), r \neq 1$

*The series $\Sigma a_n$ is said to be* **divergent**, *if the sequence $\langle S_n \rangle$ of partial sums of $\Sigma a_n$ is divergent.*

*The series $\Sigma a_n$ is said to* **oscillate**, *if the sequence $\langle S_n \rangle$ of partial sums of $\Sigma a_n$ oscillates.*

## EXAMPLES

**Example 1. (Geometric Series)**

*Show that the series* $1 + r + r^2 + r^3 + \ldots (r > 0)$ *converges if* $r < 1$ *and diverges if* $r \geq 1$. [D.U., B.Sc. (G) 1991]

**Solution.** We have

$$S_n = 1 + r + r^2 + \ldots + r^{n-1} \qquad \ldots(1)$$

or
$$S_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}.$$

**Case I.** Suppose $r < 1$. Then

$$\lim_{n\to\infty} S_n = \frac{1}{1-r} - \frac{1}{1-r}\left(\lim_{n\to\infty} r^n\right)$$

$$= \frac{1}{1-r}. \qquad \left[\because \lim_{n\to\infty} r^n = 0, \text{ if } |r| < 1.\right]$$

Thus $\langle S_n \rangle$ is a convergent sequence and so the given series is convergent.

**Case II.** Suppose $r = 1$. Then $S_n = 1 + 1 + 1 + \ldots + 1 = n$, by (1)

Clearly, the sequence $\langle S_n \rangle = \langle n \rangle = \langle 1, 2, 3, \ldots, n, \ldots \rangle$ is divergent and so the given series is divergent.

**Case III.** Suppose $r > 1$. Then $S_n > n$, by (1). Thus $\lim S_n = +\infty$ and so the given series is divergent.

**Remark.** As an application of Example 1, we see that

(*a*) $\Sigma \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$. is convergent. $\left(\because r = \frac{1}{2} < 1\right)$

(*b*) The series $1 + 1 + 1 + \ldots$ is divergent. $(\because r = 1)$

(*c*) $\Sigma 3^n = 3 + 3^2 + 3^3 + \ldots$ is divergent. $(\because r = 3 > 1)$

**Example 2.** *Show that the series* $\Sigma (-1)^{n-1}$ *oscillates.*

**Solution.** We see that

$$S_1 = 1,\ S_2 = 1 - 1 = 0,\ S_3 = 1 - 1 + 1 = 1,$$
$$S_4 = 1 - 1 + 1 - 1 = 0,\ S_5 = 1 \text{ and so on.}$$
$$\therefore \quad \langle S_n \rangle = \langle 1, 0, 1, 0, 1, 0, \ldots \rangle, \text{ which oscillates.}$$

Hence the given series oscillates.

**Example 3.** *Test for convergence the series* :

(*i*) $\sum_{n=1}^{\infty} (-1)^n n$, (*ii*) $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{3}\right)$.

[D.U., Maths (H) 1999]

**Solution.** (*i*) We see that

$$\sum_{n=1}^{\infty} (-1)^n n = -1 + 2 - 3 + 4 - 5 + 6 \ldots$$

$$\therefore \quad S_1 = -1,\ S_2 = -1 + 2 = 1,\ S_3 = -1 + 2 - 3 = -2,$$
$$S_4 = -1 + 2 - 3 + 4 = 2, \quad S_5 = -3, S_6 = 3 \text{ etc.}$$
$$\Rightarrow \quad \langle S_n \rangle = \langle -1, 1, -2, 2, -3, 3, \ldots \rangle, \text{ which is not bounded.}$$

Hence $\langle S_n \rangle$ is not convergent and so $\Sigma (-1)^n n$ is not convergent.

(*ii*) $$\sum_{n=1}^{\infty} \sin\frac{n\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + 0 - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 0 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + 0 \ldots$$

The sequence $\langle S_n \rangle$ of partial sums of the above series is

$$\langle \tfrac{\sqrt{3}}{2}, \sqrt{3}, \sqrt{3}, \tfrac{\sqrt{3}}{2}, 0, 0, \tfrac{\sqrt{3}}{2}, \sqrt{3}, \ldots \rangle$$

Clearly, $\limsup S_n = \sqrt{3}$ and $\liminf S_n = 0$.

It follows that $\langle S_n \rangle$ is not convergent. Hence the given series is not convergent.

**Example 4.** *Show that the series*

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots \textit{ is convergent.}$$

**Solution.** We have

$$S_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)}$$
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}.$$

$$\therefore \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1.$$

Hence the given series converges to 1.

**Example 5. (p-series)**

*Show that the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots (p > 0)$$

*converges if* $p > 1$ *and diverges if* $p \leq 1$.

For the solution, see Example 72 of Section 4.10.

$$W_n = (a_1 + b_1) + \dots + (a_n + b_n) = S_n + T_n \ \forall \ n \in \mathbf{N}.$$

$$\therefore \lim_{n \to \infty} T_n = \lim_{n \to \infty} W_n - \lim_{n \to \infty} S_n = w - s < \infty.$$

It follows that $\Sigma\, b_n$ is convergent, which is a contradiction. Hence $\Sigma\,(a_n + b_n)$ diverges.

## 4.3 Two Basic Theorems

**Theorem 1. (A necessary condition for convergence)**

*If the series $\Sigma\, u_n$ converges, then $\lim\limits_{n \to \infty} u_n = 0$.*

*Show, by an example, that the converse is not true.*

[D.U., Maths (H) 1996 ; B.Sc. (G) 1996 ; B.A. (P) 1992]

**Proof.** (*i*) Let $\langle S_n \rangle$ be the sequence of partial sums of the series $\Sigma\, u_n$. Then

$$S_n = u_1 + u_2 + \dots + u_{n-1} + u_n,\ S_{n-1} = u_1 + u_2 + \dots + u_{n-1}.$$

Now $\quad S_n - S_{n-1} = u_n. \qquad \dots(1)$

Since the series $\Sigma\, u_n$ converges, so $\langle S_n \rangle$ converges.

Let $\quad \lim\limits_{n \to \infty} S_n = l.$ Then $\lim\limits_{n \to \infty} S_{n-1} = l. \qquad \dots(2)$

From (1) and (2), $\quad \lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} S_n - \lim\limits_{n \to \infty} S_{n-1} = l - l = 0.$

Hence $\quad \lim\limits_{n \to \infty} u_n = 0.$

(*ii*) The converse may not be true.

The series $\quad \Sigma\, u_n = \Sigma \frac{1}{n}$ is divergent. (See Example 5 or Example 13).

But $\quad \lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} \frac{1}{n} = 0.$

Thus we see that $\lim\limits_{n \to \infty} u_n = 0$, but the series $\Sigma\, u_n = \Sigma \frac{1}{n}$ is divergent.

**Corollary.** *If $\lim\limits_{n \to \infty} u_n \neq 0$, then the series $\Sigma\, u_n$ cannot converge.*

**Proof.** Suppose $\Sigma\, u_n$ converges. Then by the above theorem, $\lim\limits_{n \to \infty} u_n = 0$, which is contrary to the given condition.

### EXAMPLES

**Example 9.** *Show that the series*

$$\sum \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots \textit{ is not convergent.}$$

**Solution.** We have $u_n = \dfrac{n}{n+1} = \dfrac{1}{1 + \frac{1}{n}}$.

**Ex. 1.** *Show that the series* $\sum\limits_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$ *diverges.*

**Hint.** Similar to Example 9.

**Ex. 2.** *Test for convergence the series* $\sum\limits_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$.

[D.U., Maths (H) 1996 ; B.Sc (G) 1999]

**Hint.** Similar to Example 11.

**Ex. 3.** *Show that the series* $\sum\limits_{n=1}^{\infty} \left(\frac{1}{n^2}\right)^{1/n}$ *is not convergent.*

**Hint.** Similar to Example 12.

**Ex. 4.** *Show that* $\lim\limits_{n \to \infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1$ *and* $\lim\limits_{n \to \infty} \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = 1.$

**Theorem 2. (Cauchy's Principle of Convergence)**

*A necessary and sufficient condition for a series* $\Sigma\, u_n$ *to converge is that to each* $\varepsilon > 0$, *there exists a positive integer m, such that*

$$| u_{m+1} + u_{m+2} + \ldots + u_n | < \varepsilon, \text{ for all } n \geq m.$$

[D.U., B.A. (P) 1996 ; B.Sc. (G) 1993]

**Proof.** Let $\langle S_n \rangle$ be the sequence of partial sums of the series $\Sigma\, u_n$. By definition, $\Sigma\, u_n$ converges $\Leftrightarrow \langle S_n \rangle$ converges

$\Leftrightarrow$ To each $\varepsilon > 0$, there exists a positive integer $m$ such that
$| S_n - S_m | < \varepsilon \; \forall \; n \geq m$
(Cauchy's Principle of Convergence for Sequences)

$\Leftrightarrow \; | (u_1 + u_2 + \ldots + u_m + u_{m+1} + u_{m+2} + \ldots + u_n) - (u_1 + u_2 + \ldots + u_m) | < \varepsilon \; \forall \; n \geq m$

$\Leftrightarrow \; | u_{m+1} + u_{m+2} + \ldots + u_n | < \varepsilon \; \forall \; n \geq m.$

Hence the theorem.

**Example 13.** *Show that the series*

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots \text{ does not converge.}$$

[D.U., B.Sc. (G) 1993]

**Solution.** Suppose the given series converges. Then the sequence $\langle S_n \rangle$ of partial sums of the given series is convergent. By Cauchy's principle of convergence for sequences, for $\varepsilon = \frac{1}{2}$, there exists a positive integer $m$ such that

$$| S_n - S_m | < \tfrac{1}{2} \; \forall \; n \geq m$$

or $$\left|\left(1+\frac{1}{2}+\ldots+\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{n}\right) - \left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right)\right| < \frac{1}{2}\ \forall\ n \geq m$$

or $$\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{n}<\frac{1}{2}\quad \forall \quad n \geq m. \qquad \ldots(1)$$

On taking $n = 2m$, we see that

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{n}=\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{2m}$$
$$>\frac{1}{2m}+\frac{1}{2m}+\ldots+\frac{1}{2m}=m\cdot\frac{1}{2m}=\frac{1}{2}.$$

$$\therefore \quad \frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{n}>\frac{1}{2}. \qquad (n=2m>m) \qquad \ldots(2)$$

Since (1) and (2) are contradictory statements, the given series does not converge.

### 4.4 Positive Term Series

*An infinite series all of whose terms are positive is called a* **positive term series.**

Consider a positive term series $\Sigma u_n$, $u_n > 0 \ \forall \ n$.

We have $\quad S_{n+1} = u_1 + u_2 + \ldots + u_n + u_{n+1} = S_n + u_{n+1}$

or $$S_{n+1} - S_n = u_{n+1} > 0 \quad \forall \quad n.$$

$\therefore$ $S_{n+1} > S_n \ \forall \ n$ and so the sequence $\langle S_n \rangle$ of partial sums of $\Sigma u_n$ is monotonically increasing.

We know that a monotonically increasing sequence is convergent if and only if it is bounded above. Hence we have the following :

**Theorem 3. (A Fundamental Test for a Positive Term Series)**

*A positive term series $\Sigma u_n$ is convergent if and only if its sequence $\langle S_n \rangle$ of partial sums is bounded above.*

*Equivalently, a positive term series $\Sigma u_n$ converges $\Leftrightarrow S_n < K \ \forall \ n$.*

*($K$ being some positive real number).* [D.U., Maths (H) 1999]

**Remarks 1.** We know that a monotonic sequence can either converge or diverge but cannot oscillate. *Hence a positive term series either converges or diverges.*

**2.** *If $\Sigma u_n$ is a positive term series such that $\lim\limits_{n \to \infty} u_n \neq 0$, then $\Sigma u_n$ diverges.*

**Proof.** By corollary of Theorem 1, the given series $\Sigma u_n$ cannot converge. Since $\Sigma u_n$ is a positive term series, so $\Sigma u_n$ diverges (by Remark 1).

**Ex.** *Define an infinite series. When is a series said to be oscillating ? Can a positive term series be oscillating ?* [D.U., Maths (H) 1999]

**Hint.** A positive term series cannot be oscillating [See Remark 1].

### 4.5 Some Comparison Tests

#### 4.5.1 First Comparison Test

*Let $\Sigma u_n$ and $\Sigma v_n$ be two positive term series such that*

$$u_n \leq k\, v_n \quad \forall \quad n \geq m. \qquad \ldots(1)$$

*($k$ being a fixed positive number and $m$ a fixed positive integer)*

*Then* (i) $\Sigma v_n$ *converges* $\Rightarrow$ $\Sigma u_n$ *converges.*

(ii) $\Sigma u_n$ *diverges* $\Rightarrow$ $\Sigma v_n$ *diverges.*

[D.U., Physics (H) 1996]

**Proof.** Let $\langle S_n \rangle$ and $\langle T_n \rangle$ be the sequences of partial sums of the series $\Sigma u_n$ and $\Sigma v_n$, respectively. For $n \geq m$, we have

$$S_n - S_m = (u_1 + u_2 + \ldots + u_m + u_{m+1} + u_{m+2} + \ldots + u_n) - (u_1 + u_2 + \ldots + u_m)$$

or $$S_n - S_m = u_{m+1} + u_{m+2} + \ldots + u_n. \qquad \ldots(2)$$

Similarly, $\quad T_n - T_m = v_{m+1} + v_{m+2} + \ldots + v_n. \qquad \ldots(3)$

From (1) and (2), we obtain

$$S_n - S_m \leq k\,(v_{m+1} + v_{m+2} + \ldots v_n)$$

or $$S_n - S_m \leq k\,(T_n - T_m), \text{ using (3)}$$

or $$S_n \leq k\, T_n + a, \qquad \ldots(4)$$

where $\quad a = S_m - k\, T_m \quad$ is a fixed number.

(i) *Suppose $\Sigma v_n$ converges.*

By Theorem 3, the sequence $\langle T_n \rangle$ of partial sums of $\Sigma v_n$ is bounded above *i.e.*, there exists a positive real number $t$ such that

$$T_n \leq t \quad \forall \quad n. \qquad \ldots(5)$$

From (4) and (5), $\quad S_n \leq kt + a \quad \forall \quad n.$

Thus the sequence $\langle S_n \rangle$ of partial sums of $\Sigma u_n$ is bounded above and so $\Sigma u_n$ is convergent (Theorem 3).

(ii) *Suppose $\Sigma u_n$ diverges.*

Then $$\lim_{n \to \infty} S_n = +\infty. \qquad \ldots(6)$$

From (4) and (6), we obtain

$$T_n \geq \frac{1}{k}(S_n - a) \Rightarrow \lim_{n \to \infty} T_n = +\infty \quad (\because k > 0).$$

It follows that the sequence $\langle T_n \rangle$ of partial sums of the series $\Sigma v_n$ diverges and so $\Sigma v_n$ diverges.

### EXAMPLES

**Example 14.** *Test for convergence the series*

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots+\frac{1}{n^n}+\ldots$$

[D.U., B.A. (P) 1999 ; B.Sc. (G) 1991, 90]

**Solution.** Clearly, $n^n > 2^n$, for $n > 2$.

$$\therefore \quad \frac{1}{n^n} < \frac{1}{2^n}, \quad \text{for} \quad n > 2.$$

Since $\Sigma \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$ is a geometric series with common ratio $\frac{1}{2} < 1$, so $\Sigma\, 1/2^n$ is convergent (Example 1). Hence by First Comparison Test, $\Sigma\, 1/n^n$ is convergent.

**Example 15.** *Show that the series* $\Sigma\, e^{-n^2}$ *converges.*

**Solution.** We know $e^x > x$, for $x > 0$. Then

$$e^{n^2} > n^2 \Rightarrow \frac{1}{e^{n^2}} < \frac{1}{n^2} \Rightarrow e^{-n^2} < \frac{1}{n^2} \quad \forall \quad n.$$

Since $\Sigma \frac{1}{n^2}$ converges, so by First Comparison Test, $\Sigma\, e^{-n^2}$ converges.

**Example 16.** *Test for convergence the series :*

(*i*) $\sum_{n=2}^{\infty} \frac{1}{\log n}$, (*ii*) $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$. [D.U., B.Sc. (G) 1998]

**Solution.** (*i*) We know $\log n < n$ for all $n \geq 2$.

$$\therefore \quad \frac{1}{\log n} > \frac{1}{n} \quad \forall \quad n \geq 2 \quad \text{or} \quad \frac{1}{n} < \frac{1}{\log n} \quad \forall \quad n \geq 2.$$

Since $\Sigma \frac{1}{n}$ diverges, so by First Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverges.

(*ii*) We know $\frac{1}{n^2 \log n} < \frac{1}{n^2}$, $\forall\, n \geq 3$.

Since $\Sigma \frac{1}{n^2}$ converges, so by First Comparison Test,

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \log n} \text{ converges.}$$

**Example 17.** *Test for convergence the series* :

(*i*) $1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \dots + \frac{1}{n\,!} + \dots$,

(*ii*) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n\,!}}$. [D.U., Maths (H) 1999]

**Solution.** We notice that $n\,! \geq 2^{n-1}$ $\forall$ $n \geq 2$.

$$\therefore \quad \frac{1}{n\,!} \leq \frac{1}{2^{n-1}} \quad \forall \quad n \geq 2.$$

Now $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$ being a geometric series with common ratio $\frac{1}{2} < 1$ is convergent. Hence, by First Comparison Test, $\Sigma \frac{1}{n\,!}$ is convergent.

(*ii*) We know $\frac{1}{\sqrt{n\,!}} \leq \frac{1}{2^{\frac{1}{2}(n-1)}}$ $\forall$ $n \geq 2$.

The series on the R.H.S. being a geometric series with common ratio $\frac{1}{\sqrt{2}} < 1$ is convergent. Hence $\Sigma \frac{1}{\sqrt{n\,!}}$ is convergent.

### 4.5.2 Second Comparison Test

*If* $\Sigma\, u_n$ *and* $\Sigma\, v_n$ *are two positive term series such that*

$$\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}} \quad \forall \quad n \geq m, \qquad \dots(1)$$

*then* (*i*) $\Sigma\, v_n$ *converges* $\Rightarrow$ $\Sigma\, u_n$ *converges,*

(*ii*) $\Sigma\, u_n$ *diverges* $\Rightarrow$ $\Sigma\, v_n$ *diverges.*

**Proof.** For $n \geq m$, we have

$$\frac{u_m}{u_n} = \frac{u_m}{u_{m+1}} \cdot \frac{u_{m+1}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+3}} \cdots\cdots \frac{u_{n-1}}{u_n}$$

$$\geq \frac{v_m}{v_{m+1}} \cdot \frac{v_{m+1}}{v_{m+2}} \cdot \frac{v_{m+2}}{v_{m+3}} \cdots\cdots \frac{v_{n-1}}{v_n}, \text{ by (1)}$$

$$= \frac{v_m}{v_n}.$$

$$\text{Thus} \quad \frac{u_m}{u_n} \geq \frac{v_m}{v_n} \quad \text{or} \quad \frac{u_n}{u_m} \leq \frac{v_n}{v_m} \quad \forall \quad n \geq m \qquad \dots(2)$$

$$\therefore \quad u_n \leq k\, v_n \quad \forall \quad n \geq m,$$

where $k = u_m/v_m$ is a fixed positive number.

Applying First Comparison Test in (2), we obtain

(*i*) $\Sigma\, v_n$ converges $\Rightarrow$ $\Sigma\, u_n$ converges,

(*ii*) $\Sigma\, u_n$ diverges $\Rightarrow$ $\Sigma\, v_n$ diverges.

### 4.5.3 (Limit Form Test)

*Let* $\Sigma\, u_n$ *and* $\Sigma\, v_n$ *be two positive term series such that*

$$\lim_{n \to \infty} \frac{u_n}{v_n} = l, \text{ (} l \text{ is finite and non-zero).}$$

*Then* $\Sigma\, u_n$ *and* $\Sigma\, v_n$ *converge or diverge together.*

[D.U., B.A. (P) 1991]

**Proof.** Since $\frac{u_n}{v_n} > 0$ for all $n$, so $\lim_{n \to \infty} \frac{u_n}{v_n} \geq 0$.

Thus $l > 0$, as $l \neq 0$.
Let $\varepsilon > 0$ be some number such that $l - \varepsilon > 0$.

Since $\lim_{n \to \infty} \frac{u_n}{v_n} = l$, so there exists a positive integer $m$ such that

$$\left| \frac{u_n}{v_n} - l \right| < \varepsilon \quad \forall \quad n \geq m$$

or

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \quad \forall \quad n \geq m, \quad \text{where } v_n > 0 \ \forall \ n.$$

$$\therefore \qquad (l - \varepsilon)\, v_n < u_n < (l + \varepsilon)\, v_n \quad \forall \quad n \geq m. \qquad \text{...(1)}$$

From (1), we consider

$$u_n < (l + \varepsilon)\, v_n \quad \forall \quad n \geq m. \qquad \text{...(2)}$$

(Here $l + \varepsilon$ is a fixed positive number)

Applying First Comparison Test in (2), we obtain

$$\left.\begin{array}{l} \Sigma v_n \text{ converges} \Rightarrow \Sigma u_n \text{ converges,} \\ \Sigma u_n \text{ diverges} \Rightarrow \Sigma v_n \text{ diverges.} \end{array}\right\} \qquad \text{...(A)}$$

From (1), we consider

$$(l - \varepsilon)\, v_n < u_n \quad \forall \quad n \geq m$$

or

$$v_n < k u_n \quad \forall \quad n \geq m, \qquad \text{...(3)}$$

where $k = \frac{1}{l - \varepsilon} > 0$ is a fixed number.

Applying First Comparison Test in (3), we obtain

$$\left.\begin{array}{l} \Sigma u_n \text{ converges} \Rightarrow \Sigma v_n \text{ converges,} \\ \Sigma v_n \text{ diverges} \Rightarrow \Sigma u_n \text{ diverges.} \end{array}\right\} \qquad \text{...(B)}$$

From (A) and (B), it follows that

$$\Sigma u_n \text{ converges} \Leftrightarrow \Sigma v_n \text{ converges,}$$
$$\Sigma u_n \text{ diverges} \Leftrightarrow \Sigma v_n \text{ diverges.}$$

Hence the two series $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together.

**Remark 1.** *If $l = 0$ or $l = \infty$, then the conclusion of the above test may not hold good.* [D.U., B.A. (P) 1991]

(*i*) Let $\Sigma u_n = \Sigma \frac{1}{n^2}$ and $\Sigma v_n = \Sigma \frac{1}{n}$.

Then $\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n}{n^2} = \lim_{n \to \infty} \frac{1}{n} = 0.$

*Thus $l = 0$ and in this case $\Sigma u_n$ converges and $\Sigma v_n$ diverges.*

(*ii*) Let $\Sigma u_n = \Sigma \frac{1}{n}$ and $\Sigma v_n = \Sigma \frac{1}{n^2}$.

Then $\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n^2}{n} = \lim_{n \to \infty} n = \infty.$

*Thus $l = \infty$ and in this case $\Sigma u_n$ diverges and $\Sigma v_n$ converges.*

**Remark 2.** In order to apply the *Limit Form Test* to a given series $\Sigma u_n$, we have to select a series $\Sigma v_n$ (which is usually a $p$-series) in which the $n$th term $v_n$ behaves as $u_n$, for large values of $n$. This is symbolically written as $u_n \sim v_n$. Thus, for large values of $n$, we see that

$$\frac{1}{n^2 + 1} \sim \frac{1}{n^2}, \quad \frac{1}{\sqrt{n} + \sqrt{n+1}} \sim \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}} \text{ etc.}$$

**Remark 3.** We recall that $\Sigma \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $0 < p \leq 1$.

## EXAMPLES

**Example 18.** *Test each of the following series for convergence :*

(*i*) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots$

(*ii*) $\frac{1}{\sqrt{1.2}} + \frac{1}{\sqrt{2.3}} + \frac{1}{\sqrt{3.4}} + \ldots$

**Solution.** (*i*) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots$

The $n$th term of this series of $u_n = \frac{2n - 1}{n(n+1)(n+2)}$.

For large values of $n$, $u_n \sim \frac{2n}{n.n.n} = \frac{2}{n^2}$.

Let $v_n = \frac{1}{n^2}$. Then $\frac{u_n}{v_n} = \frac{(2n-1)\, n^2}{n(n+1)(n+2)}$

or

$$\frac{u_n}{v_n} = \left(\frac{n}{n+1}\right)\left(\frac{2n-1}{n+2}\right) = \frac{1}{1 + (1/n)} \cdot \frac{2 - (1/n)}{1 + (2/n)}.$$

Now $\lim_{n \to \infty} \frac{u_n}{v_n} = 2 \neq 0$ and finite.

So $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together (Limit Form Test). Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ converges, so $\Sigma u_n$ converges.

(*ii*) $\frac{1}{\sqrt{1.2}} + \frac{1}{\sqrt{2.3}} + \frac{1}{\sqrt{3.4}} + \ldots$

The $n$th term is $u_n = \dfrac{1}{\sqrt{n}\,(n+1)} \sim \dfrac{1}{\sqrt{n}.n} = \dfrac{1}{n}$.

Let $v_n = \dfrac{1}{n}$. Then $\dfrac{u_n}{v_n} = \sqrt{\dfrac{n}{n+1}} = \dfrac{1}{\sqrt{1+(1/n)}}$.

Now $\lim\limits_{n \to \infty} \dfrac{u_n}{v_n} = 1 \neq 0$ and finite.

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n = \Sigma \dfrac{1}{n}$ diverges, so $\Sigma\, u_n$ diverges.

**Example 19.** *Test for convergence the series*

$$\frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \ldots$$

**Solution.** We have

$$u_n = \frac{1}{\sqrt{n+1}+\sqrt{n+2}} \sim \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}}.$$

Let $v_n = \dfrac{1}{\sqrt{n}}$. Then $\dfrac{u_n}{v_n} = \dfrac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$.

Now $\lim\limits_{n \to \infty} \dfrac{u_n}{v_n} = \lim\limits_{n \to \infty} \dfrac{1}{\sqrt{1+(1/n)}+\sqrt{1+(2/n)}} = \dfrac{1}{\sqrt{2}}$.

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n = \Sigma \dfrac{1}{\sqrt{n}}$ diverges, so $\Sigma\, u_n$ diverges.

**Example 20.** *Test for convergence the series*

$$\frac{1}{4.6} + \frac{\sqrt{3}}{6.8} + \frac{\sqrt{5}}{8.10} + \ldots$$

[D.U., B.A. (P) 1999]

**Solution.** $u_n = \dfrac{\sqrt{2n-1}}{(2n+2)(2n+4)} \sim \dfrac{\sqrt{2n}}{2n\,.\,2n}$.

Let $\quad v_n = \dfrac{\sqrt{n}}{n\,.\,n} = \dfrac{1}{n^{3/2}}$. Then

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \sqrt{\frac{2n-1}{n}} \left(\frac{n}{2n+2}\right)\left(\frac{n}{2n+4}\right)$$

$$= \lim_{n\to\infty} \sqrt{2-(1/n)}\, \frac{1}{\{2+(2/n)\}}\, \frac{1}{\{2+(4/n)\}}$$

$$= \frac{\sqrt{2}}{2.2} = \frac{1}{2\sqrt{2}} \neq 0 \text{ and finite.}$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n = \Sigma \dfrac{1}{n^{3/2}}$ converges, so the given series converges.

**Example 21.** *Test for converges the series*

$$\frac{1}{3.7} + \frac{1}{4.9} + \frac{1}{5.11} + \frac{1}{6.13} + \ldots$$

[D.U., B.A. (P) 1994 ; B.Sc. (G) 1991]

**Solution.** We have

$$u_n = \frac{1}{(n+2)(2n+5)} \sim \frac{1}{n.2n} = \frac{1}{2n^2}.$$

Let $\quad v_n = \dfrac{1}{n^2}$. Then

$$\frac{u_n}{v_n} = \frac{n}{n+2} \cdot \frac{n}{2n+5} = \frac{1}{1+(2/n)} \cdot \frac{1}{2+(5/n)}.$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2} \neq 0 \text{ and finite.}$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n = \Sigma\, 1/n^2$ converges, so the given series $\Sigma\, u_n$ also converges.

**Example 22.** *Test for convergence the series whose nth term is*

$$\frac{\sqrt{n+1}-\sqrt{n-1}}{n}.$$

[D.U., B.Sc. (G) 1992]

**Solution.**

$$u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} = \frac{(\sqrt{n+1}-\sqrt{n-1})(\sqrt{n+1}+\sqrt{n-1})}{n\,(\sqrt{n+1}+\sqrt{n-1})}$$

or

$$u_n = \frac{(n+1)-(n-1)}{n\,(\sqrt{n+1}+\sqrt{n-1})}$$

$$= \frac{2}{n\,(\sqrt{n+1}+\sqrt{n-1})} \sim \frac{2}{n\,(\sqrt{n}+\sqrt{n})} = \frac{1}{n\sqrt{n}}.$$

Let $\quad v_n = \dfrac{1}{n\sqrt{n}} = \dfrac{1}{n^{3/2}}$, so that $\Sigma\, v_n$ converges.

Now $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{2\sqrt{n}}{(\sqrt{n+1}+\sqrt{n-1})}$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{1+(1/n)}+\sqrt{1-(1/n)}} = 1 \neq 0.$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n$ converges, so the given series $\Sigma\, u_n$ also converges.

**Example 23.** *Show that* $\Sigma \dfrac{1}{\sqrt{n}+\sqrt{n+1}}$ *diverges.*

**Hint.** Take $v_n = \dfrac{1}{\sqrt{n}}$. Then $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{1}{2}$.

**Example 24.** *Test for convergence the series whose nth term is* $(\sqrt{n+1}-\sqrt{n})$. [D.U., Physics (H) 1993]

**Hint.** Rationalise to get Example 23.

**Example 25.** *Test for convergence the series :*

$(i)\ \sum_{n=1}^{\infty}(\sqrt{n^3+1}-\sqrt{n^3})$, $\quad (ii)\ \sum_{n=1}^{\infty}(\sqrt{n^4+1}-\sqrt{n^4-1})$.

[D.U., Maths (H) 1993, B.Sc. (G) 1998 ; Physics (H) 1999]

**Solution.**

(*i*) $$u_n=\sqrt{n^3+1}-\sqrt{n^3}=\frac{(\sqrt{n^3+1}-\sqrt{n^3})(\sqrt{n^3+1}+\sqrt{n^3})}{\sqrt{n^3+1}+\sqrt{n^3}}$$

or $$u_n=\frac{(n^3+1)-n^3}{\sqrt{n^3+1}+\sqrt{n^3}}=\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}\sim\frac{1}{2\sqrt{n^3}}.$$

Let $v_n=\frac{1}{\sqrt{n^3}}=\frac{1}{n^{3/2}}$, so that $\Sigma v_n$ converges.

Now $$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\sqrt{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}}$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{1+(1/n)^3}+1}=\frac{1}{2}\neq 0 \text{ and finite.}$$

So $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together.

Since $\Sigma v_n$ converges, so the given series $\Sigma u_n$ converges.

(*ii*) Let $u_n=\sqrt{n^4+1}-\sqrt{n^4-1}$. Now

$$u_n=\frac{(\sqrt{n^4+1}-\sqrt{n^4-1})(\sqrt{n^4+1}+\sqrt{n^4-1})}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$=\frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

or $$u_n=\frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}\sim\frac{2}{\sqrt{n^4}+\sqrt{n^4}}=\frac{2}{2n^2}=\frac{1}{n^2}.$$

Let $v_n=\frac{1}{n^2}\left(=\frac{1}{\sqrt{n^4}}\right)$, so that $\Sigma v_n$ converges.

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{2\sqrt{n^4}}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$=\lim_{n\to\infty}\frac{2}{\sqrt{1+(1/n)^4}+\sqrt{1-(1/n)^4}}=1\neq 0 \text{ and finite.}$$

So $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together.

Since $\Sigma v_n$ converges, so $\Sigma u_n$ converges.

**Example 26.** (*a*) *Show that* $\Sigma(\sqrt{n+1}-\sqrt{n})$ *diverges.*

(*b*) *Show that* $\Sigma(\sqrt{n^2+1}-n)$ *diverges.*

(*c*) *Show that* $\Sigma(\sqrt{n^4+1}-n^2)$ *converges.*

[D.U., B.A. (P) 1995 ; Physics (H) 1994]

[**Hint.** Rationalise and proceed like Example 25].

**Example 27.** *Test for convergence the series whose nth term is* $\{(n^3+1)^{1/3}-n\}$.

[D.U., Maths (H) 1996 ; B.Sc. (G) 1995 ; Physics (H) 1997]

**Solution.** Let $u_n=(n^3+1)^{1/3}-n=n\left[\left(1+\frac{1}{n^3}\right)^{1/3}-1\right]$

or $$u_n=n\left[\left(1+\frac{1}{3n^3}-\frac{1}{9n^6}+\dots\right)-1\right]$$

or $$u_n=\frac{1}{3n^2}-\frac{1}{9n^5}+\dots$$

Let $v_n=\frac{1}{n^2}$, so that $\Sigma v_n$ converges.

Now $\lim_{n\to\infty}\frac{u_n}{v_n}=\frac{1}{3}\neq 0$ and finite.

So $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together.

Since $\Sigma v_n$ converges, so $\Sigma u_n$ converges.

**Example 28.** *Test for convergence the series :*

(*i*) $\sum \sin\frac{1}{n}$, $\quad$ (*ii*) $\sum \sin\frac{1}{n^2}$, $\quad$ (*iii*) $\sum\frac{1}{\sqrt{n}}\tan\frac{1}{n}$.

**Solution.** (*i*) Let $u_n=\sin\left(\frac{1}{n}\right)$ and $v_n=\frac{1}{n}$.

Now $$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\sin(1/n)}{1/n}=1\neq 0 \text{ and finite.}$$

So $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together.

Since $\Sigma v_n=\Sigma\frac{1}{n}$ diverges, so $\Sigma\sin\frac{1}{n}$ diverges.

(*ii*) Let $u_n=\sin\frac{1}{n^2}$ and $v_n=\frac{1}{n^2}$.

Now $$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\sin(1/n^2)}{1/n^2}=1\neq 0 \text{ and finite.}$$

So $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together.

Since $\Sigma v_n=\Sigma\frac{1}{n^2}$ converges, so $\Sigma\sin\frac{1}{n^2}$ converges.

(*iii*) Let $u_n = \frac{1}{\sqrt{n}} \tan \frac{1}{n}$ and $v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$.

Now $\lim\limits_{n \to \infty} \frac{u_n}{v_n} = \lim\limits_{n \to \infty} \frac{\tan (1/n)}{1/n} = 1 \neq 0$ and finite.

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n = \Sigma \frac{1}{n^{3/2}}$ converges, so $\Sigma\, u_n$ converges.

**Example 29.** *Test for convergence the series* :

(*a*) $\sum \frac{n+1}{n^p}$. (*b*) $\sum \frac{1}{(2n-1)^p}$.

**Solution.** (*a*) Let $u_n = \frac{n+1}{n^p}$ and $v_n = \frac{n}{n^p} = \frac{1}{n^{p-1}}$.

$$\therefore \quad \lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n+1}{n} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1.$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Now $\Sigma\, v_n = \Sigma \frac{1}{n^{p-1}}$ converges, if $p - 1 > 1$ *i.e.*, $p > 2$ ;

and diverges if $p - 1 \leq 1$ *i.e.*, $p \leq 2$.

Hence $\sum \frac{n+1}{n^p}$ converges if $p > 2$ and diverges if $p \leq 2$.

(*b*) Let $u_n = \frac{1}{(2n-1)^p}$ and $v_n = \frac{1}{n^p}$.

$$\therefore \quad \lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \left(\frac{n}{2n-1}\right)^p = \lim_{n \to \infty} \left(\frac{1}{2-(1/n)}\right)^p = \frac{1}{2^p}.$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Hence $\Sigma \frac{1}{(2n-1)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

**Example 30.** *Test for convergence the series*

$$\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \ldots$$

**Solution.** The *n*th term of the given series is

$$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1}.$$

Let $$v_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}.$$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \left(\sqrt{\frac{n+1}{n}} - \frac{1}{\sqrt{n}}\right) \lim_{n \to \infty} \frac{1}{\left[\left(\left(\frac{n+2}{n}\right)^3 - \frac{1}{n^3}\right)\right]}$$

$$= \lim_{n \to \infty} \sqrt{1 + \left(\frac{1}{n}\right)} \cdot \lim_{n \to \infty} \frac{1}{\left[1 + \left(\frac{2}{n}\right)\right]^3} = 1.$$

So $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n = \Sigma \frac{1}{n^{5/2}}$ converges, so the given series converges.

**Example 31.** *Test the convergence of the series* $\Sigma \frac{1}{n^{1+(1/n)}}$.

**Solution.** We have $u_n = \frac{1}{n^{1+(1/n)}}$. Let $v_n = \frac{1}{n}$.

Then $$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{1}{n^{1/n}} = 1.$$

Thus the two series $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together. Since $\Sigma\, v_n = \Sigma\, 1/n$ diverges, so the given series $\Sigma\, u_n$ diverges.

**Example 32.** *Show that the series* $\frac{1.2}{3^2.4^2} + \frac{3.4}{5^2.6^2} + \frac{5.6}{7^2.8^2} + \ldots$ *converges.*

[D.U., Physics (H) 1998]

**Solution.** The *n*th term of the given series is

$$u_n = \frac{(2n-1)\,.\,2n}{(2n+1)^2\,(2n+2)^2} \sim \frac{1}{n^2}. \quad \text{Let } v_n = \frac{1}{n^2}. \text{ Then}$$

$$\frac{u_n}{v_n} = \frac{2n^3\,(2n-1)}{(2n+1)^2\,(2n+2)^2} = \frac{4-(2/n)}{(2+1/n)^2\,(2+2/n)^2}.$$

$\therefore \lim\limits_{n \to \infty} \frac{u_n}{v_n} = 1$, so that $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Since $\Sigma\, v_n = \Sigma\, 1/n^2$ converges, so the given series $\Sigma\, u_n$ also converges.

**Example 33.** *Show that the series* $1 + a + b + a^2 + b^2 + a^3 + b^3 + \ldots,$ $0 < a < b < 1$ *is convergent.*

**Solution.** Since $0 < a < b < 1$, $a^n < b^n\ \forall\ n \in \mathbf{N}$. Further, if $u_n$ is the *n*th term of the given series, then $u_n \leq b^n\ \forall\ n \geq 0$.

Since $\Sigma\, b^n$ is a geometric series with common ratio $b, 0 < b < 1$ ; $\Sigma\, b^n$ is convergent. Hence, by First comparison test, the given series $\Sigma\, u_n$ is convergent.

**Example 34.** *Test for convergence the series* $\sum\limits_{n=1}^{\infty} \frac{1}{3^n + x}$, *for all positive values of x.*

**Solution.** Clearly $3^n + x > 3^n$, since $x > 0$

$$\Rightarrow \quad \frac{1}{3^n + x} < \frac{1}{3^n} \quad \forall\, n \in \mathbf{N}.$$

Now $\Sigma \frac{1}{3^n}$ being a geometric series with common ratio $\frac{1}{3} < 1$ is convergent. Hence, by First Comparison Test, the given series is convergent for all $x > 0$.

**Example 35.** *Let $\Sigma\, a_n$ be a convergent series of positive terms. Prove that $\Sigma\, a_n^2$ is convergent. Show by an example that the converse may not be true.*

**Solution.** We have $a_n > 0 \;\forall\; n$. Since $\Sigma\, a_n$ is convergent, $\lim\limits_{n \to \infty} a_n = 0$.

It follows that for $\varepsilon = 1$, there exists some $m \in \mathbf{N}$ such that

$$|\, a_n - 0\,| < 1 \text{ or } a_n < 1 \;\; \forall \;\; n \geq m \;\; \Rightarrow \;\; 0 \leq a_n^2 < a_n \;\; \forall \;\; n \geq m.$$

Since $\Sigma\, a_n$ is convergent, so by First comparison test, $\Sigma\, a_n^2$ is convergent.

If $\Sigma\, a_n^2$ is convergent, then $\Sigma\, a_n$ may not be convergent.

Let $\Sigma\, a_n = \Sigma\, 1/n$, so that $\Sigma\, a_n^2 = \Sigma\, 1/n^2$.

We see that $\Sigma\, a_n^2$ is convergent, but $\Sigma\, a_n$ is not convergent.

**Example 36.** *Let $\Sigma\, a_n$ be a convergent series of positive terms, prove that $\Sigma \frac{a_n}{n}$ is convergent.*

**Solution.** By Example 35, $\Sigma\, a_n^2$ is convergent. We have

$$\left(a_n - \frac{1}{n}\right)^2 \geq 0 \;\; \forall \;\; n \;\; \Rightarrow \;\; a_n^2 + \frac{1}{n^2} \geq 2\,\frac{a_n}{n} \quad \forall \;\; n$$

or

$$\frac{a_n}{n} \leq \frac{1}{2}\left(a_n^2 + \frac{1}{n^2}\right) \quad \forall \;\; n.$$

Since $\Sigma\, a_n^2$ and $\Sigma \frac{1}{n^2}$ are convergent, so $\Sigma \frac{1}{2}\left(a_n^2 + \frac{1}{n^2}\right)$ is convergent.

Hence, by First comparison test, $\Sigma \frac{a_n}{n}$ is convergent.

**Example 37.** *If $\Sigma\, a_n$ is a convergent series of positive terms, prove that $\frac{a_n}{1 + a_n}$ is convergent.*

**Solution.** Since $\Sigma\, a_n$ is convergent, $\lim\limits_{n \to \infty} a_n = 0$.

Let $b_n = \frac{a_n}{1 + a_n} \;\forall\; n$. Then $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = \lim\limits_{n \to \infty} (1 + a_n) = 1 + 0 = 1$.

Hence $\Sigma\, a_n$ and $\Sigma\, b_n$ converge or diverge together. Since $\Sigma\, a_n$ is convernt, so $\Sigma\, b_n = \Sigma \frac{a_n}{1 + a_n}$ is convergent.

## EXERCISES

Test each of the following series for convergence :

**1.** $\sum\limits_{n=1}^{\infty} \frac{1}{2n - 1}$. **(Ans.** Divergent)

$\left[\textbf{Hint. } u_n = \frac{1}{2n - 1},\; v_n = \frac{1}{n}\right]$

**2.** $\sum\limits_{n=1}^{\infty} \frac{n^2}{n^3 + 4}$. **(Ans.** Divergent)

$\left[\textbf{Hint. } v_n = \frac{1}{n}\right]$

**3.** $\sum\limits_{n=1}^{\infty} \frac{1}{2^n + x}$, for all positive values of $x$. **(Ans.** Convergent)

[**Hint.** Proceed like Example 34]

**4.** $\frac{1}{1 + 2} + \frac{1}{1 + 2^2} + \frac{1}{1 + 2^3} + \ldots$ **(Ans.** Convergent)

**5.** $\sum\limits_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{1}{(n + 1)^2}\right]$ **(Ans.** Convergent)

$\left[\textbf{Hint. } \frac{1}{n^2} + \frac{1}{(n + 1)^2} < \frac{1}{n^2} + \frac{1}{n^2} = 2\left(\frac{1}{n^2}\right) \;\forall\; n\right]$

**6.** $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \ldots$ **(Ans.** Divergent)

$\left[\textbf{Hint. } u_n = \frac{\sqrt{n}}{2n + 3},\; v_n = \frac{1}{\sqrt{n}}\right]$

**7.** $\frac{1}{1.3.5} + \frac{1}{2.4.6} + \frac{1}{3.5.7} + \frac{1}{4.6.8} + \ldots$ **(Ans.** Convergent)

$\left[\textbf{Hint. } u_n = \frac{1}{n\,(n + 2)\,(n + 4)},\; v_n = \frac{1}{n^3}\right]$

**8.** $\frac{1}{\sqrt{3.4}} + \frac{1}{\sqrt{4.5}} + \frac{1}{\sqrt{5.6}} + \frac{1}{\sqrt{6.7}} + \ldots$ **(Ans.** Divergent)

$\left[\textbf{Hint. } u_n = \frac{1}{\sqrt{(n + 2)\,(n + 3)}},\; v_n = \frac{1}{n}\right]$

**9.** $\frac{1}{1.2^2} + \frac{1}{2.3^2} + \frac{1}{3.4^2} + \ldots$ **(Ans.** Convergent)

$\left[\textbf{Hint. } u_n = \frac{1}{n\,.\,(n + 1)^2}. \text{ Let } v_n = \frac{1}{n^3}.\right]$

**Case III.** We shall give examples of two series : one convergent and the other divergent, but both satisfying $\lim\limits_{n\to\infty} (u_n)^{1/n} = 1$.

The series $\Sigma\, u_n = \Sigma \frac{1}{n}$ is divergent, but

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = 1.$$

The series $\Sigma\, u_n = \Sigma \frac{1}{n^2}$ is convergent, but

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = 1.$$

## EXAMPLES

**Example 38.** *Test for convergence the series whose nth term is*

$$\frac{n^{n^2}}{(n+1)^{n^2}}.$$ [D.U., B.Sc. (G) 1995, 94, 92]

**Solution.** We have $u_n = \left[\dfrac{n}{n+1}\right]^{n^2}$

$$\Rightarrow \qquad (u_n)^{1/n} = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n}.$$

$$\therefore \qquad \lim_{n\to\infty} (u_n)^{1/n} = \frac{1}{\lim\limits_{n\to\infty} (1+1/n)^n} = \frac{1}{e} < 1 \,.\ (e = 2.718).$$

Hence, by Cauchy's root test, the given series is convergent.

**Example 39.** *Show that the series* $\sum\limits_{n=1}^{\infty} (n^{1/n} - 1)^n$ *converges.*

**Solution.** Let $u_n = (n^{1/n} - 1)^n$, so that $(u_n)^{1/n} = n^{1/n} - 1$.

Now $\lim\limits_{n\to\infty} (u_n)^{1/n} = \lim\limits_{n\to\infty} n^{1/n} - 1 = 1 - 1 = 0 < 1$.

Hence, by Cauchy's root test, the given series converges.

**Example 40.** *Test for the convergence the series whose nth term is*

$$\left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$$ [D.U., Maths (H) 1992]

**Solution.** We have $u_n = \dfrac{1}{\left(1 + \dfrac{1}{\sqrt{n}}\right)^{n^{3/2}}}$. Then

**Ex. 3.** *Test for convergence the series*

$$(i)\ \sum_{n=2}^{\infty} \frac{1}{(\log n)^n}, \qquad (ii)\ \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

**Hint.** (*i*) $\lim\limits_{n \to \infty} (u_n)^{1/n} = \lim\limits_{n \to \infty} \frac{1}{\log n} = 0 < 1$. Hence the given series is convergent.

(*ii*) $\lim\limits_{n \to \infty} (u_n)^{1/n} = \lim\limits_{n \to \infty} \frac{1}{n} = 0 < 1$, so the given series is convergent.

## 4.7 D'Alembert's Ratio Test

*Let* $\Sigma u_n$ *be a positive term series such that*

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = l. \qquad ...(1)$$

*Then* (*i*) $\Sigma u_n$ *converges if* $l > 1$,

(*ii*) $\Sigma u_n$ *diverges if* $l < 1$.

*Test fails if* $l = 1$. [D.U., B.Sc. (G) 1991 ; Maths (H) 1997, 95 ; Physics (H) 1998 ; B.A. (P) 1999, 96]

(not confirmed)

**Proof. Case I.** Let $l > 1$.

We can choose some $\varepsilon > 0$ such that $l - \varepsilon > 1$ or $\alpha > 1, \alpha = l - \varepsilon$.

Using (1), there exists a positive integer $m_1$ such that

$$\left| \frac{u_n}{u_{n+1}} - l \right| < \varepsilon \quad \forall \quad n \geq m_1$$

or

$$l - \varepsilon < \frac{u_n}{u_{n+1}} < l + \varepsilon \quad \forall \quad n \geq m_1.$$

Consider $\frac{u_n}{u_{n+1}} > (l - \varepsilon) = \alpha \quad \forall \quad n \geq m_1$

or

$$\frac{u_n}{u_{n+1}} > \frac{\alpha^{n+1}}{\alpha^n} \quad \text{or} \quad \frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \quad \forall \quad n \geq m_1, \qquad ...(2)$$

where $\Sigma v_n = \Sigma \frac{1}{\alpha^n} = \frac{1}{\alpha} + \frac{1}{\alpha^2} + ...$, being a geometric series with common ratio $\frac{1}{\alpha} < 1$, is convergent. Hence $\Sigma u_n$ is convergent (by Second Comparison Test as applied in (2)).

**Case II.** Let $l < 1$.

We can choose another $\varepsilon > 0$ such that $l + \varepsilon < 1$

or

$$\beta < 1, \ \beta = l + \varepsilon.$$

**Proof.** It is given that the sequence $\left\{\dfrac{u_n}{u_{n+1}}\right\}$ diverges to ∞.

So there exists a positive integer $m$ such that

$$\frac{u_n}{u_{n+1}} > 2 \quad \forall \quad n \geq m$$

or $$\frac{u_n}{u_{n+1}} > \frac{2^{n+1}}{2^n} \text{ or } \frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \quad \forall \quad n \geq m,$$

where $\Sigma\, v_n = \Sigma \dfrac{1}{2^n}$, being a geometric series with common ratio $\dfrac{1}{2} < 1$, is convergent. Hence, by Second Comparison Test, $\Sigma\, u_n$ is convergent.

**Remark 3.** *If $\Sigma\, u_n$ is a positive term series such that*

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = 0, \textit{ then } \Sigma\, u_n \textit{ is convergent.}$$

**Proof.** For $\varepsilon = \dfrac{1}{2}$, there exists a positive integer $m$ such that

$$\left| \frac{u_{n+1}}{u_n} - 0 \right| < \frac{1}{2} \quad \forall \quad n \geq m \quad \text{or} \quad \frac{u_n}{u_{n+1}} > 2 \quad \forall \quad n \geq m.$$

The result now follows by Remark 2.

## EXAMPLES

**Example 43.** *Test for convergence the series*

$$\frac{1}{2} + \frac{1}{2.2^2} + \frac{1}{3.2^3} + \frac{1}{4.2^4} + \ldots$$

**Solution.** We have

$$u_n = \frac{1}{n.2^n},\ u_{n+1} = \frac{1}{(n+1)\, 2^{n+1}}.$$

$$\therefore \quad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{2\,(n+1)}{n} = 2 \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 2 > 1.$$

By Ratio Test, the given series converges.

**Example 44.** *Test for convergence the series*

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \ldots$$

**Solution.** We have

$$u_n = \left[\frac{1.2.3 \ldots n}{3.5.7 \ldots (2n+1)}\right]^2,\ u_{n+1} = \left[\frac{1.2.3 \ldots. n\,(n+1)}{3.5.7 \ldots (2n+1)\,(2n+3)}\right]^2$$

**Example 47.** *Test for convergence the series* $\sum_{n=1}^{\infty} \frac{r^n}{n!}$, *where r is any positive number.*

**Solution.** We have

$$u_n = \frac{r^n}{n!},\ u_{n+1} = \frac{r^{n+1}}{(n+1)!}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{1}{r}\frac{(n+1)!}{n!},$$

and $$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{r} \lim_{n \to \infty} (n+1) = \infty. \qquad [\because r > 0].$$

By Ratio Test, the given series converges.

**Example 48.** *Test for convergence the series whose nth term is* $r^n/n^n$, $r > 0$.

**Solution.** We have $u_n = \frac{r^n}{n^n}$, $u_{n+1} = \frac{r^{n+1}}{(n+1)^{n+1}}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{r}\frac{(n+1)^{n+1}}{n^n} = \frac{1}{r}\left(\frac{n+1}{n}\right)^n (n+1)$$

$$= \frac{1}{r}\left(1 + \frac{1}{n}\right)^n (n+1).$$

Now $$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{r} \cdot e \cdot \infty = \infty.$$

By Ratio Test, the given series converges.

**Example 49.** *Test for convergence the series* $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

**Solution.** We have

$$u_n = \frac{n!}{n^n},\ u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}. \text{ Now}$$

$$\frac{u_n}{u_{n+1}} = \frac{n!}{(n+1)!}\frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{1}{(n+1)}\left(\frac{n+1}{n}\right)^n (n+1) = \left(1 + \frac{1}{n}\right)^n.$$

$$\therefore \quad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e = 2.718 > 1.$$

By Ratio Test, the given series converges.

Let $v_n = \frac{1}{n\sqrt{n}}$, so that $\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$ converges.

$$\text{Now} \lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n}{n+1} \sqrt{\frac{n}{n+2}}$$

$$= \lim_{n \to \infty} \frac{1}{\left(1 + \frac{1}{n}\right)\sqrt{1 + \frac{2}{n}}} = 1.$$

So $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together. Since $\Sigma v_n$ converges, therefore $\Sigma u_n$ also converges (for $x = 1$).

Hence the given series converges for $x \le 1$ and diverges for $x > 1$.

**Example 52.** *Test for convergence the series :*

(*i*) $\frac{x}{\sqrt{5}} + \frac{x^3}{\sqrt{7}} + \frac{x^5}{\sqrt{9}} + \ldots \qquad (x > 0)$

(*ii*) $\frac{x^2}{2\sqrt{1}} + \frac{x^3}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \ldots \qquad (x > 0)$

[D.U., Physics (H) 1995 ; B.A. (P) 1995]

(*iii*) $\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \ldots \qquad (x > 0)$

**Solution.** (*i*) $u_n = \frac{x^{2n-1}}{\sqrt{2n+3}}$, $u_{n+1} = \frac{x^{2n+1}}{\sqrt{2n+5}}$.

$$\therefore \quad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2} \lim_{n \to \infty} \sqrt{\frac{2n+5}{2n+3}} = \frac{1}{x^2}.$$

By Ratio Test, $\Sigma u_n$ converges if $1/x^2 > 1$ *i.e.*, $x^2 < 1$ *i.e.*, $x < 1$ and diverges if $x^2 > 1$ *i.e.*, $x > 1$. The test fails if $x = 1$.

$$\text{For } x = 1, \qquad u_n = \frac{1}{\sqrt{2n+3}} \sim \frac{1}{\sqrt{2n}}.$$

$$\text{Let} \qquad v_n = \frac{1}{\sqrt{n}}.$$

$$\text{Then} \quad \lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \sqrt{\frac{n}{2n+3}} = \frac{1}{2}.$$

Since $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$ diverges, so $\Sigma u_n$ diverges (for $x = 1$).

Hence the given series converges if $x < 1$ and diverges if $x \ge 1$.

(*ii*) We have $u_n = \frac{x^{n+1}}{(n+1)\sqrt{n}}$, $u_{n+1} = \frac{x^{n+2}}{(n+2)\sqrt{n+1}}$.

$$\therefore \quad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x} \lim_{n \to \infty} \left(\frac{n+2}{n+1}\right)\sqrt{\frac{n+1}{n}} = \frac{1}{x}.$$

(*ii*) We have $u_n = \dfrac{n^n x^n}{n\,!}$ and $u_{n+1} = \dfrac{n^{n+1} x^{n+1}}{(n+1)\,!}$.

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \left(\frac{n+1}{n}\right)\cdot\frac{1}{x} = \left(1+\frac{1}{n}\right)\frac{1}{x} \text{ and } \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

By Ratio test, $\Sigma u_n$ converges if $1/x > 1$ *i.e.*, $x < 1$ and diverges if $1/x < 1$ *i.e.*, $x > 1$. For $x = 1$, we have

$$u_n = \frac{n^n}{n\,!} \text{ and } u_{n+1} = \frac{(n+1)^{n+1}}{(n+1)\,!}.$$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} \Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{e} < 1.$$

So the given series diverges for $x = 1$. Hence the given series converges if $x < 1$ and diverges if $x \geq 1$

**Example 54.** *Test the series :*

$$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \dots$$

*for convergence for all positive values of x.*

[D.U., B.A. (P) 1997 ; B.Sc. (G) 1991]

**Solution.** Ignoring the first term of the series, we have

$$u_n = \frac{x^{2n}}{2n} \text{ and } u_{n+1} = \frac{x^{2n+2}}{2n+2}.$$

$$\therefore \qquad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1}{x^2}\cdot\frac{2n+2}{2n}$$

$$= \lim_{n\to\infty}\frac{1}{x^2}\left(1+\frac{1}{n}\right) = \frac{1}{x^2}.$$

By Ratio Test, $\Sigma u_n$ converges if $1/x^2 > 1 \Rightarrow x < 1$ ($\because x > 0$) and $\Sigma u_n$ diverges if $x > 1$. Ratio Test fails if $x = 1$.

For $\qquad x = 1$, $u_n = \dfrac{1}{2n}$. Let $v_n = \dfrac{1}{n}$.

$\therefore \lim\limits_{n\to\infty}\dfrac{u_n}{v_n} = \dfrac{1}{2}$. Since $\Sigma v_n = \Sigma\dfrac{1}{n}$ diverges, so $\Sigma u_n$ diverges for $x = 1$.

Hence $\Sigma u_n$ converges if $x < 1$ and diverges if $x \geq 1$.

**Example 55.** *Test for convergence the series :*

$$1 + \frac{x}{2} + \frac{x^2}{5} + \dots + \frac{x^n}{n^2+1} + \dots$$

*for all positive values of x.*

[D.U., B.A. (P) 1994 ; B.Sc. (G) 1990]

**Solution.** Ignoring the first term of the series, we have

$$u_n = \frac{x^n}{n^2+1},\ u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x} \cdot \frac{(n+1)^2+1}{n^2+1} = \frac{1}{x}.$$

By Ratio Test, $\Sigma u_n$ converges if $1/x > 1$ *i.e.*, $x < 1$ and $\Sigma u_n$ diverges if $x > 1$. Ratio Test fails if $x = 1$.

For $x = 1$, $\quad u_n = \dfrac{1}{n^2+1}$.

Clearly, $\dfrac{1}{n^2+1} < \dfrac{1}{n^2} \quad \forall \quad n \in \mathbf{N}$.

Since $\Sigma \dfrac{1}{n^2}$ converges, so by First Comparison Test, $\Sigma \dfrac{1}{n^2+1}$ converges.

Hence $\Sigma u_n$ converges for $x \le 1$ and diverges for $x > 1$.

**Example 56.** *Test for convergence the series with nth term* :

(*i*) $\dfrac{\sqrt{n}\, x^n}{\sqrt{n^2+1}}$ $\quad (x > 0)$, [D.U., Physics (H) 1994 ; B.A. (P) 1991]

(*ii*) $\sqrt{\dfrac{n-1}{n^3+1}}\, x^n$ $\quad (x > 0)$.

**Solution.** (*i*) $\quad u_n = \dfrac{\sqrt{n}}{\sqrt{n^2+1}} x^n,\ u_{n+1} = \dfrac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} x^{n+1}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x} \sqrt{\frac{n}{n+1}} \sqrt{\frac{(n+1)^2+1}{n^2+1}}$$

$$= \frac{1}{x\sqrt{1+1/n}} \sqrt{\frac{(1+1/n)^2+1/n^2}{1+1/n^2}}.$$

$\therefore \lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = \dfrac{1}{x}$. By Ratio Test, the given series converges if $1/x > 1$ *i.e.*, $x < 1$ and diverges if $x > 1$. The test fails if $x = 1$.

For $\quad x = 1,\ u_n = \dfrac{\sqrt{n}}{\sqrt{n^2+1}} \sim \dfrac{\sqrt{n}}{\sqrt{n^2}}$.

Let $\quad v_n = \dfrac{\sqrt{n}}{\sqrt{n^2}} = \dfrac{1}{\sqrt{n}}$, so that $\Sigma v_n$ diverges.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = 1.$$

So $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together.

Since $\Sigma v_n$ diverges, so $\Sigma u_n$ diverges (for $x = 1$).

Hence the given series converges if $x < 1$ and diverges if $x \ge 1$.

(*ii*) $u_n = \sqrt{\dfrac{n-1}{n^3+1}}\, x^n,\ u_{n+1} = \sqrt{\dfrac{n}{(n+1)^3+1}}\, x^{n+1}$.

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x} \sqrt{\frac{n-1}{n}} \sqrt{\frac{(n+1)^3+1}{n^3+1}} = \frac{1}{x}.$$

By Ratio Test, the given series converges if $1/x > 1$ *i.e.*, $x < 1$ and diverges if $1/x < 1$ *i.e.*, $x > 1$. The test fails if $x = 1$.

For $x = 1$, $\quad u_n = \sqrt{\dfrac{n-1}{n^3+1}} \sim \sqrt{\dfrac{n}{n^3}} = \dfrac{1}{n}$.

Let $\quad v_n = \dfrac{\sqrt{n}}{\sqrt{n^3}} = \dfrac{1}{n}$, so that $\Sigma v_n$ diverges.

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \sqrt{\frac{n-1}{n}} \sqrt{\frac{n^3}{n^3+1}} = 1.$$

So $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together.

Since $\Sigma v_n = \Sigma \dfrac{1}{n}$ diverges, so $\Sigma u_n$ diverges (for $x = 1$).

Hence the given series converges if $x < 1$ and diverges if $x \ge 1$.

**Example 57.** *Show that* $\dfrac{1}{1^p} + \dfrac{2}{2^p} + \dfrac{3}{3^p} + \ldots + \dfrac{n}{n^p} + \ldots$ *converges if* $p > 2$ *and diverges if* $p \le 2$. [D.U., Maths (H) 1994]

**Solution.** The given series is $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{p-1}}$, which converges if $p - 1 > 1$ *i.e.*, $p > 2$ and diverges if $p - 1 \le 1$ *i.e.*, $p \le 2$.

**Example 58.** *Test for convergence the series*

$$1 + \frac{x^2}{2^p} + \frac{x^4}{4^p} + \frac{x^6}{6^p} + \ldots$$

**Solution.** Ignoring the first term of the given series, the *n*th term is

$$u_n = \frac{x^{2n}}{(2n)^p} \text{ and } u_{n+1} = \frac{x^{2n+2}}{(2n+2)^p}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2} \lim_{n\to\infty} \left(\frac{2n+2}{2n}\right)^p = \frac{1}{x^2}.$$

By Ratio Test, the given series converges if $1/x^2 > 1$ *i.e.*, $x^2 < 1$ *i.e.*, $|x| < 1$ and diverges if $|x| > 1$. The test fails if $|x| = 1$.

For $\quad |x| = 1 \quad$ or $\quad x^2 = 1,\ u_n = \dfrac{1}{(2n)^p} = \dfrac{1}{2^p \cdot n^p}$.

Let $v_n = \dfrac{1}{n^p}$.

Now $\lim\limits_{n \to \infty} \dfrac{u_n}{v_n} = \dfrac{1}{2^p} \neq 0$ and finite.

So $\Sigma u_n$ and $\Sigma v_n = \Sigma \dfrac{1}{n^p}$ converge or diverge together.

Hence for $|x| = 1$, $\Sigma u_n$ converges if $p > 1$ and diverges if $p \leq 1$.

**Example 59.** *Test for convergence the series*

$$1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \dots \qquad (p > 0).$$

**Solution.** We have

$$u_n = \frac{n^p}{n!} \text{ and } u_{n+1} = \frac{(n+1)^p}{(n+1)!}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n^p}{(n+1)^p} \cdot \frac{(n+1)!}{n!} = (n+1) \cdot \left(\frac{n}{n+1}\right)^p = \frac{n+1}{(1+1/n)^p}.$$

$\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} = \infty$. Hence $\Sigma u_n$ is convergent, by Ratio Test.

**Example 60.** *Test for convergence the series* :

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \frac{30}{33}x^4 + \dots$$

[D.U., Maths (H) 1998 ; Physics (H) 1997]

**Solution.** Ignoring the first term, we have

$$u_n = \frac{2^{n+1} - 2}{2^{n+1} + 1} x^n, \quad u_{n+1} = \frac{2^{n+2} - 2}{2^{n+2} + 1} x^{n+1}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \frac{2^{n+1} - 2}{2^{n+2} - 2} \cdot \frac{2^{n+2} + 1}{2^{n+1} + 1} = \frac{1}{x} \cdot \frac{2^n - 1}{2^{n+1} - 1} \cdot \frac{2^{n+2} + 1}{2^{n+1} + 1}$$

$$= \frac{1}{x} \cdot \frac{1 - (1/2)^n}{2 - (1/2)^n} \cdot \frac{2 + (1/2)^{n+1}}{1 + (1/2)^{n+1}}.$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}\left(\frac{1-0}{2-0}\right)\left(\frac{2+0}{1+0}\right) = \frac{1}{x}.$$

By Ratio Test, $\Sigma u_n$ converges if $1/x > 1$ *i.e.*, $x < 1$ and diverges if $x > 1$. Ratio Test fails if $x = 1$.

For $x = 1$, $\quad \lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} \dfrac{2^{n+1} - 2}{2^{n+1} + 1} = \lim\limits_{n \to \infty} \dfrac{1 - (1/2)^n}{1 + (1/2)^{n+1}}$

*i.e.*, $\quad \lim\limits_{n \to \infty} u_n = 1 \neq 0$. So $\Sigma u_n$ diverges for $x = 1$.

**Hence $\Sigma u_n$ converges if $x < 1$ and diverges if $x \geq 1$.**

## EXERCISES

Test for convergence the series :

1. $\dfrac{1}{3} + \dfrac{1.2}{3.5} + \dfrac{1.2.3}{3.5.7} + \dots$ (**Ans.** Convergent)

2. $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \dots$ (**Ans.** Convergent)

3. $\dfrac{2!}{3} + \dfrac{3!}{3^2} + \dfrac{4!}{3^3} + \dots + \dfrac{(n+1)!}{3^n} + \dots$ [D.U., B.Sc. (G) 1997] (**Ans.** Divergent)

   [**Hint.** Similar to Example 50]

4. $1 + \dfrac{3}{2!} + \dfrac{5}{3!} + \dfrac{7}{4!} + \dots$ (**Ans.** Convergent)

5. $\sum\limits_{n=1}^{\infty} \dfrac{n+2}{2^n + 5}$. (**Ans.** Convergent)

   [**Hint.** Similar to Example 45 (*ii*)]

6. $\dfrac{1}{2} + \dfrac{2!}{2^3} + \dfrac{3!}{2^5} + \dots$ (**Ans.** Divergent)

7. $1 + \dfrac{2!}{2^2}x + \dfrac{3!}{2^3}x^2 + \dots (x > 0)$ (**Ans.** Divergent)

8. $\sum\limits_{n=1}^{\infty} \dfrac{5^n}{n^2 + 5}$. (**Ans.** Divergent)

   [**Hint.** Let $u_n = \dfrac{5^n}{n^2 + 5}$. Then $u_{n+1} = \dfrac{5^{n+1}}{(n+1)^2 + 5}$.

   $\therefore \quad \dfrac{u_n}{u_{n+1}} = \dfrac{5^n}{5^{n+1}} \cdot \dfrac{(n+1)^2 + 5}{n^2 + 5} = \dfrac{1}{5} \cdot \dfrac{(1+1/n)^2 + 5/n^2}{1 + 5/n^2}$.

   $\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} = \dfrac{1}{5} < 1$. Hence, by Ratio Test, $\Sigma u_n$ diverges.]

9. $\dfrac{x}{2} + \dfrac{x^2}{5} + \dfrac{x^3}{10} + \dfrac{x^4}{17} + \dots$

   (**Ans.** Convergent for $x \leq 1$ and divergent for $x > 1$)

   [**Hint.** Similar to Example 55]

10. $1 + 3x + 5x^2 + 7x^3 + \dots (x > 0)$

    (**Ans.** Convergent for $x < 1$ and divergent for $x \geq 1$)

    [**Hint.** $u_n = (2n-1)x^{n-1}$, $u_{n+1} = (2n+1)x^n$.

    For $x = 1$, $\Sigma u_n = 1 + 3 + 5 + 7 + \dots$ is divergent.]

### 4.8. Raabe's Test

*Let $\Sigma u_n$ be a positive term series such that*

$$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l. \qquad \dots(1)$$

**Case III.** Consider the two series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$, where the first series is divergent and the second series is convergent [See Example 73], but for both the series

$$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1.$$

**Remark.** *Raabe's Test is stronger than Ratio Test.*

*When Ratio test fails i.e., when lim* $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = 1$, *Raabe's test may be applied.*

## EXAMPLES

**Example 61.** *Test for convergence the series :*

$$\sum_{n=1}^{\infty} \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} \cdot \frac{1}{n}.$$

**Solution.** We have

$$u_n = \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} \cdot \frac{1}{n},$$

$$u_{n+1} = \frac{1.3.5 \ldots (2n-1)(2n+1)}{2.4.6 \ldots 2n(2n+2)} \cdot \frac{1}{n+1}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)(n+1)}{(2n+1)n} = \left(1 + \frac{1}{n}\right)\frac{(1+1/n)}{(1+1/2n)}$$

Now $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = 1$ means that the Ratio test fails.

So we apply Raabe's Test. We have

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(n+1)(2n+2)}{n(2n+1)} - 1\right] = \frac{3n+2}{2n+1} = \frac{3+\frac{2}{n}}{2+\frac{1}{n}}.$$

$$\therefore \quad \lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{3}{2} > 1.$$

Hence, by Raabe's test, the given series converges.

**Example 62.** ***Test for convergence the series :***

(*i*) $\sum_{n=1}^{\infty} \frac{2.4.6....2n}{1.3.5.....(2n+1)}$, [D.U., B.A. (P) 1999]

(*ii*) $\sum_{n=1}^{\infty} \frac{1.3.5...(2n-1)}{2.4.6....2n} \cdot \frac{4n+1}{2n+3}$.

(*ii*) $u_n = \dfrac{2.4 \ldots (2n+2)}{3.5 \ldots (2n+3)} x^{n-1}$,

$$u_{n+1} = \frac{2.4 \ldots (2n+2)\,(2n+4)}{3.5 \ldots (2n+3)\,(2n+5)} x^n .$$

$$\therefore \quad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x} \lim_{n \to \infty} \frac{2n+5}{2n+4} = \frac{1}{x}.$$

By Ratio Test, $\Sigma u_n$ converges if $1/x > 1$ *i.e.*, $x < 1$ and diverges if $x > 1$. The test fails if $x = 1$.

For $x = 1$, $\dfrac{u_n}{u_{n+1}} = \dfrac{2n+5}{2n+4}$. Now

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{2n+5}{2n+4} - 1\right) = \frac{n}{2n+4} = \frac{1}{2+(4/n)}.$$

$$\therefore \lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{1}{2} < 1.$$

By Raabe's Test, $\Sigma u_n$ diverges (for $x = 1$).

Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

**Example 64.** *Test for convergence the series* :

$$1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \ldots \; (x > 0) \qquad \text{[D.U., Physics (H) 1995]}$$

[**Hint.** Similar to Example 63(*i*).]

**Example 65.** *Examine the convergence of the following series* :

(*i*) $\displaystyle\sum_{n=1}^{\infty} \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} \cdot \frac{x^{2n+1}}{2n+1}, \; (x > 0).$

[D.U., B.Sc. (G) 1992, B.A. (P) 1996]

(*ii*) $\displaystyle\sum_{n=1}^{\infty} \frac{1.3.5 \ldots (4n-5)\,(4n-3)}{2.4.6 \ldots (4n-4)\,(4n-2)} \cdot \frac{x^{2n}}{4n}, \; (x > 0)$

[D.U., Maths (H) 1993, 91 ; B.A. (P) 1998]

**Solution.** (*i*) We have

$$u_n = \frac{1.3 \ldots (2n-1)}{2.4 \ldots 2n} \cdot \frac{x^{2n+1}}{2n+1},$$

$$u_{n+1} = \frac{1.3 \ldots (2n-1)\,(2n+1)}{2.4 \ldots 2n\,(2n+2)} \cdot \frac{x^{2n+3}}{2n+3}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x^2} \frac{(2n+2)\,(2n+3)}{(2n+1)\,(2n+1)}, \quad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}.$$

**By Ratio test, $\Sigma u_n$ converges** if $1/x^2 > 1$ *i.e.*, $x < 1$ (as $x > 0$) and **diverges if $x > 1$. The test fails** for $x = 1$.

For $x = 1$, $\quad \dfrac{u_n}{u_{n+1}} = \dfrac{(2n+2)(2n+3)}{(2n+1)^2}$. Now

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(2n+2)(2n+3)}{(2n+1)^2} - 1\right] = \frac{n(6n+2)}{(2n+1)^2}.$$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{6 + 2/n}{(2 + 1/n)^2} = \frac{6}{4} > 1.$$

By Raabe's test, $\Sigma u_n$ converges (for $x = 1$).

Hence the given series converges for $x \le 1$ and diverges for $x > 1$.

(*ii*) $\quad u_n = \dfrac{1.3 \ldots (4n-3)}{2.4 \ldots (4n-2)} \cdot \dfrac{x^{2n}}{4n}$,

$$u_{n+1} = \frac{1.3 \ldots (4n-3)(4n+1)}{2.4 \ldots (4n-2)(4n+2)} \frac{x^{2n+2}}{(4n+4)}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x^2} \frac{(4n+2)(4n+4)}{4n(4n+1)}, \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}.$$

By Ratio test, $\Sigma u_n$ converges if $1/x^2 > 1$ *i.e.*, $x^2 < 1$ *i.e.*, $x < 1$ (as $x > 0$) and diverges if $x > 1$. The test fails if $x = 1$.

For $x = 1$, $\quad \dfrac{u_n}{u_{n+1}} = \dfrac{(4n+2)(4n+4)}{4n(4n+1)}$. Now

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(4n+2)(4n+4)}{4n(4n+1)} - 1\right] = \frac{20n+8}{4(4n+1)},$$

and $\quad \displaystyle\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{20}{16} > 1.$

y Raabe's Test, $\Sigma u_n$ converges (for $x = 1$).

ence the given series converges for $x \le 1$ and diverges for $x > 1$.

**xample 66.** *Discuss the convergence of the series :*

$$\sum_{n=1}^{\infty} \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} x^n, \; x > 0. \qquad \text{[D.U., B.Sc. (G) 1996]}$$

**Solution.** We have

$$u_n = \frac{1.3.5 \ldots\ldots (2n-1)}{2.4.6 \ldots\ldots 2n} x^n, \quad u_{n+1} = \frac{1.3.5 \ldots\ldots (2n+1)}{2.4.6 \ldots (2n+2)} x^{n+1}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \frac{2n+2}{2n+1} = \frac{1}{x} \text{ as } n \to \infty.$$

Thus $\Sigma u_n$ converges if $1/x > 1$ i.e., $x < 1$ and $\Sigma u_n$ diverges if $x > 1$. Ratio test fails if $x = 1$. For $x = 1$, $n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = \dfrac{1}{2} < 1$ and so by Raabe's test, $\Sigma u_n$ diverges if $x = 1$. Hence $\Sigma u_n$ converges if $x < 1$ and diverges if $x \ge 1$.

**Example 67.** *Discuss the convergence of* $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$, $x > 0$.

[D.U., B.Sc. (G) 1994]

**Solution.** We have

$$u_n = \frac{(2n)!}{(n!)^2} x^n, \quad u_{n+1} = \frac{(2n+2)!}{\{(n+1)!\}^2} x^{n+1}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)^2}{(2n+2)(2n+1)} \cdot \frac{1}{x} = \frac{(n+1)}{2x(2n+1)} = \frac{1}{4x} \quad \text{as} \quad n \to \infty.$$

By Ratio test, $\Sigma u_n$ converges if $\frac{1}{4x} > 1$ i.e., $x < \frac{1}{4}$ and $\Sigma u_n$ diverges if $x > \frac{1}{4}$. For $x = \frac{1}{4}$, we have

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{2(n+1)}{(2n+1)} - 1\right] = \frac{n}{2n+1} = \frac{1}{2} < 1 \text{ as } n \to \infty.$$

By Raabe's Test, $\Sigma u_n$ diverges if $x = \frac{1}{4}$.

Hence $\Sigma u_n$ converges if $x < \frac{1}{4}$ and diverges if $x \ge \frac{1}{4}$.

**Example 68.** *Test for converges the series whose nth term is*

$$\frac{(n!)^2}{(2n)!} x^n.$$

**Solution.** We have

$$u_n = \frac{(n!)^2}{(2n)!} x^n, \; u_{n+1} = \frac{[(n+1)!]^2}{(2n+2)!} x^{n+1}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \frac{(2n+2)(2n+1)}{(n+1)^2} = \frac{2(2n+1)}{n+1} \cdot \frac{1}{x},$$

so that $\quad \displaystyle\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{4}{x}.$

By Ratio Test, $\Sigma u_n$ converges if $4/x > 1$ *i.e.*, $x < 4$ and diverges if $x > 4$. When $x = 4$, Ratio test fails and we apply Raabe's test.

For $x = 4$, $\quad \dfrac{u_n}{u_{n+1}} = \dfrac{2n+1}{2(n+1)}$.

**Solution.** On ignoring the first term, we have

$$u_n = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{1.2.3\ldots n},\ u_{n+1} = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)(\alpha+n)}{1.2.3\ldots n(n+1)}.$$

$$\therefore\quad \frac{u_n}{u_{n+1}} = \frac{n+1}{\alpha+n} \Rightarrow \frac{1+(1/n)}{1+(\alpha/n)} \Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1.$$

Thus Ratio test fails. We shall now apply Raabe's test.

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{n+1}{\alpha+n} - 1\right) = \frac{n(1-\alpha)}{\alpha+n} = (1-\alpha)\cdot\frac{1}{1+(\alpha/n)}$$

$$\therefore\ \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1-\alpha < 1 \quad (\because\ \alpha > 0)$$

Hence, the given series is divergent.

## EXERCISES

1. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}\cdot\frac{1}{2n+1}.$$ [D.U., B.Sc. (G) 1993]

**(Ans. Convergent)**

[**Hint.** See Example 65 (*i*), when $x = 1$].

2. Show that the series

$$\sum_{n=1}^{\infty} \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\frac{x^{2n}}{2n}\ (x > 0)$$

converges if $x \le 1$ and diverges if $x > 1$.

[**Hint.** Similar to Example 65 (*i*)].

3. Test the convergence of the series

$$\frac{1}{2}x + \frac{1.3}{2.4}\cdot\frac{x^2}{2} + \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{3} + \ldots$$ [D.U., Maths (H) 1990]

**(Ans. Converges if $x \le 1$, diverges if $x > 1$)**

$$\left[\textbf{Hint. } u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}\frac{x^n}{n}.\right]$$

4. Discuss the convergence of the series

$$1 + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \ldots$$

**(Ans. Converges if $\beta > \alpha + 1$, diverges if $\beta \le \alpha + 1$)**

[**Hint.** $\dfrac{u_n}{u_{n+1}} = \dfrac{n+1+\beta}{n+1+\alpha}$, on ignoring the first term.

Now proceed like Example 69.]

5. Test for convergence the series :

$$\frac{\alpha}{\beta}+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)}+\dots$$ [D.U., Physics (H) 1999, 98]

(**Ans.** Converges if $\beta > \alpha + 1$, diverges if $\beta \le \alpha + 1$)

$\left[\textbf{Hint. } \dfrac{u_n}{u_{n+1}} = \dfrac{n+\beta}{n+\alpha}\text{. Now proceed like Example 69.}\right]$

6. Test for convergence the series

$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\dots, x>0.$$

(**Ans.** Converges for $x \le 1$ and diverges for $x > 1$)

[**Hint.** Same as Example 65(*i*), after ignoring the first term.]

7. Test for convergence the series :

$$1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1.3}{2.4}\cdot\frac{1}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{7}+\dots$$

[**Hint.** $u_n = \dfrac{1.3.5\dots(2n-1)}{2.4.6\dots 2n}\cdot\dfrac{1}{2n+1}$, after omitting the first term. Now see Ex. 1. above ]

8. Show that the series

$$\frac{a}{b}+\frac{a(a+c)}{b(b+c)}+\frac{a(a+c)(a+2c)}{b(b+c)(b+2c)}+\dots (a, b, c > 0)$$

is convergent if $b > a + c$ and divergent if $b \le a + c$.

[**Hint.** Similar to Example 69.]

### 9 Convergence of the Infinite Integral $\int_1^{\infty} u(x)\,dx$.

**Definition 1.** *The infinite integral* $\int_1^{\infty} f(x)\,dx$ *is said to be* **convergent** ivergent), *if* $\lim_{t\to\infty}\int_1^t f(x)\,dx$ *is finite (infinite).*

We state a result without proof :

**Theorem 4.9.** *If* $f(x) \ge 0 \;\forall\; x \ge 1$, *then* $\int_1^{\infty} f(x)\,dx$ *is convergent iff there* xists *a positive number k such that for all* $t \ge 1$, $\int_1^t f(x)\,dx \le k$.

**Example 71.** *Examine the convergence and divergence of*

(*i*) $\displaystyle\int_1^{\infty}\frac{dx}{x}$, (*ii*) $\displaystyle\int_1^{\infty}\frac{dx}{1+x^2}$.

**Solution.** (*i*) We have

$$\int_1^t \frac{dx}{x} = \Big|\log x\Big|_1^t = \log t.$$

$$\therefore \quad \lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{t\to\infty}(\log t) = \infty. \qquad \text{Hence } \int_1^{\infty}\frac{dx}{x} \text{ diverges.}$$

(*ii*) Now $$\int_1^t \frac{dx}{1+x^2} = \Big|\tan^{-1}x\Big|_1^t = \tan^{-1}t - \frac{\pi}{4}.$$

$$\therefore \quad \lim_{t\to\infty}\int_1^t \frac{dx}{1+x^2} = \lim_{n\to\infty}\left(\tan^{-1}t-\frac{\pi}{4}\right)=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$$

Hence $\displaystyle\int_1^{\infty}\frac{dx}{1+x^2}$ converges.

**Definition 2.** *Let* $f(x)$ *be a real valued function with domain* $[1, \infty[$.

*The function* $f(x)$ *is said to be* **non-negative,** *if*

$$f(x) \ge 0 \;\forall\; x \ge 1.$$

*The function* $f(x)$ *is said to be* **monotonically decreasing,** *if*

$$x \le y \Rightarrow f(x) \ge f(y)\,;\, x, y \in [1, \infty[.$$

For example, $f(x) = 1/x^2$ is non-negative and monotonically decreasing $\forall\; x \ge 1$.

### 4.10 Cauchy's Integral Test

*If* $u(x)$ *is a non-negative, monotonically decreasing and integrable function such that* $u(n) = u_n \;\forall\; n \in \mathbf{N}$, *then the series* $\sum_{n=1}^{\infty} u_n$ *is convergent if and only if* $\int_1^{\infty} u(x)\,dx$ *is convergent.* [D.U., B.A. (P) 1995]

**Proof.** Since $u(x)$ is monotonically decreasing, so

$$u(n) \ge u(x) \ge u(n+1), \quad \text{when} \quad n \le x \le n+1.$$

Since $u(x)$ is non-negative and integrable, so

From (2) and (4), $S_n - u_1 \le \int_1^n u(x)\, dx \le k \quad \forall \; n$

or $S_n \le k + u_1 \; \forall \; n$, so that $\langle S_n \rangle$ is bounded above.

Hence the series $\Sigma\, u_n$ is convergent.

## EXAMPLES

**Example 72.** *Show that the series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p > 0)$ *is convergent if* $p > 1$ *and divergent if* $p \le 1$. [D.U., Maths (H) 1998 ; Physics (H) 1993]

**Solution.** Let $u_n = \frac{1}{n^p}$ and $u(x) = \frac{1}{x^p}$, so that $u(n) = u_n \quad \forall \; n \in \mathbf{N}$.

Clearly, for $x \ge 1$, $u(x)$ is non-negative, integrable and a decreasing function of $x$. Now

$$\int_1^t u(x)\, dx = \int_1^t x^{-p}\, dx = \frac{1}{(-p+1)} \left| x^{-p+1} \right|_1^t, \text{ if } p \ne 1,$$

$$= \left| \log x \right|_1^t, \text{ if } p = 1.$$

$$\therefore \qquad \int_1^t u(x)\, dx = \frac{1}{1-p}(t^{1-p} - 1), \text{ if } p \ne 1,$$

$$= (\log t - \log 1), \text{ if } p = 1.$$

$$\text{Now} \quad \lim_{t \to \infty} \int_1^t u(x)\, dx = \begin{cases} \dfrac{1}{1-p}\left[\lim\limits_{t \to \infty} t^{1-p} - 1\right], & \text{if } p \ne 1 \\ \lim\limits_{t \to \infty} \log t & , \text{ if } p = 1 \end{cases}$$

$$(\because \log 1 = 0)$$

$$= \begin{cases} \infty & , \text{ if } p < 1, \\ \dfrac{1}{1-p}(0-1) = \dfrac{1}{p-1}, & \text{if } p > 1, \\ \infty & , \text{ if } p = 1. \end{cases}$$

$$\text{So} \qquad \lim_{t \to \infty} \int_1^t u(x) = \infty, \text{ if } p \le 1 \text{ and } \frac{1}{p-1}, \text{ if } p > 1.$$

If follows that the improper integral $\int_1^\infty u(x)$ is convergent if $p > 1$ and divergent if $p \le 1$. Hence, by Cauchy's integral test, $\sum_{n=1}^\infty u_n = \sum_{n=1}^\infty \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

**Example 73.** *Show that the series* $\sum_{n=2}^\infty \frac{1}{n(\log n)^p}$ *is convergent if* $p > 1$ *and divergent if* $0 < p \le 1$. [D.U., Maths (H) 1998]

**Solution.** Let $u(x) = \frac{1}{x(\log x)^p}$, $(p > 0)$.

Then for $x \ge 2$, $u(x)$ is non-negative, monotonically decreasing and integrable function. Also $u(n) = u_n \ \forall \ n \in \mathbf{N}$.

$$\text{Consider } \int_2^t u(x)\,dx = \int_2^t \frac{dx}{x(\log x)^p}$$

$$= \frac{1}{-p+1} \Big| (\log x)^{-p+1} \Big|_2^t, \text{ if } p \neq 1$$

$$= \Big| \log(\log x) \Big|_2^t, \text{ if } p = 1$$

or

$$\int_2^t u(x)\,dx = \begin{cases} \frac{1}{1-p}[(\log t)^{1-p} - (\log 2)^{1-p}], & \text{if } p \neq 1, \\ [\log(\log t) - \log(\log 2)], & \text{if } p = 1. \end{cases}$$

$$\therefore \lim_{t\to\infty} \int_2^t u(x)\,dx = \begin{cases} \infty & , \text{ if } p < 1, \\ \frac{1}{p-1}(\log 2)^{1-p} & , \text{ if } p > 1, \\ \infty & , \text{ if } p = 1. \end{cases}$$

Thus $\int_2^\infty u(x)\,dx$ is convergent if $p > 1$ and divergent if $0 < p \le 1$. Hence, by Cauchy's Integral Test, the given series is convergent if $p > 1$ and divergent if $0 < p \le 1$.

**Example 74.** *Test for convergence the series* $\sum_{n=2}^\infty \frac{1}{n \log n}$. [D.U., Maths (H) 1993]

**Solution.** See Example 73 and consider the case $p = 1$.

We know $\lim_{t\to\infty} \int_2^t u(x)\,dx = \lim_{t\to\infty} [\log(\log t) - \log(\log 2)] = \infty$.

**Hence the given series diverges.**

**Example 75.** *Apply Cauchy's Integral Test to examine the convergence of the following series :*

(*i*) $\sum_{n=1}^\infty \frac{1}{n^2+1}$, (*ii*) $\sum_{n=1}^\infty \frac{1}{n^2+n}$. [D.U., Physics (H) 1994]

**Solution.** (*i*) Let $u(x) = \frac{1}{x^2+1}$, so that $u(n) = u_n \ \forall \ n \in \mathbf{N}$.

For $x \ge 1$, $u(x)$ is non-negative, monotonically decreasing and integrable function. Now

$$\lim_{t\to\infty} \int_1^t \frac{dx}{x^2+1} = \lim_{t\to\infty} \Big| \tan^{-1} x \Big|_1^\infty = \lim_{n\to\infty} \left( \tan^{-1} t - \frac{\pi}{4} \right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}, \text{ which is finite.}$$

Thus $\int_{n=1}^\infty \frac{dx}{x^2+1}$ is convergent and so $\sum_{n=1}^\infty \frac{1}{n^2+1}$ is convergent.

(By Cauchy's Integral Test)

(*ii*) Let $u(x) = \frac{1}{x^2+x}$, so that $u(n) = u_n \ \forall \ n \in \mathbf{N}$.

For $x \ge 1$, $u(x)$ is non-negative, monotonically decreasing and integrable function. Now

$$\int_1^t \frac{dx}{x^2+x} = \int_1^t \frac{dx}{x(x+1)} = \int_1^t \left( \frac{1}{x} - \frac{1}{x+1} \right) dx$$

$$= \Big| \log x - \log(x+1) \Big|_1^t = \Big| \log \frac{x}{1+x} \Big|_1^t$$

$$= \log \frac{t}{1+t} - \log \frac{1}{2} = \log \frac{1}{1+(1/t)} - \log \frac{1}{2}.$$

$$\therefore \lim_{t\to\infty} \int_1^t \frac{dx}{x^2+x} = \log 1 + \log 2 = \log 2, \text{ which is finite.}$$

Thus $\int_1^\infty u(x)\,dx$ is a convergent and so by Cauchy's Integral Test, the given series is convergent.

2. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\dots$

3. $\frac{1}{\log 2}-\frac{1}{\log 3}+\frac{1}{\log 4}-\frac{1}{\log 5}+\dots$

## 4.12 Leibnitz Test

*If an alternating series* $\sum\limits_{n=1}^{\infty}(-1)^{n-1}u_n$ *satisfies*

(*i*) $u_{n+1}\le u_n\ \forall\ n,$

(*ii*) $\lim\limits_{n\to\infty} u_n=0,$

*then the series* $\Sigma(-1)^{n-1}u_n$ *converges.*

[D.U., Maths (H) 1999, 96, 92 ; Physics (H) 1995 ;
B.Sc. (G) 1998, 90 ; B.A. (P) 1999]

**Proof.** In order to prove that the given series converges, we shall show that its sequence $\langle S_n\rangle$ of partial sums converges. For this we shall first prove that the subsequences $\langle S_{2n}\rangle$ and $\langle S_{2n+1}\rangle$ both converge to the same limit.

Now $\quad S_{2n}=u_1-u_2+u_3-u_4+\dots+u_{2n-1}-u_{2n},$

and $\quad S_{2n+2}=u_1-u_2+u_3-u_4+\dots+u_{2n-1}-u_{2n}+u_{2n+1}-u_{2n+2}.$

$\therefore\quad S_{2n+2}-S_{2n}=u_{2n+1}-u_{2n+2}\ge 0.\qquad (\because\ u_{n+1}\le u_n\ \forall\ n)$

Thus $\langle S_{2n}\rangle$ is a monotonically increasing sequence.

Again $\quad S_{2n}=u_1-u_2+u_3-u_4+u_5-u_6+\dots+u_{2n-1}-u_{2n}$

$$=u_1-[(u_2-u_3)+(u_4-u_5)+\dots+(u_{2n-2}-u_{2n-1})+u_{2n}].$$

Now each term within the bracket is positive, since

$$u_{n+1}\le u_n\quad \forall\ n\quad \text{and}\quad u_{2n}>0.$$

$\therefore\quad S_{2n}<u_1\quad \forall\ n\quad$ and $\quad$ so $\langle S_{2n}\rangle$ is bounded above.

Since $\langle S_{2n}\rangle$ is monotonically increasing **and** bounded above, therefore $\langle S_{2n}\rangle$ is convergent. ...(*iii*)

Let $\quad \lim\limits_{n\to\infty} S_{2n}=l.$

We shall now show that $\langle S_{2n+1}\rangle$ converges to $l$. We have

$$S_{2n+1}=u_1-u_2+u_3-u_4+\dots+u_{2n-1}-u_{2n}+u_{2n+1}$$

or $\quad S_{2n+1}=S_{2n}+u_{2n+1}\ \Rightarrow\ \lim\limits_{n\to\infty} S_{2n+1}=\lim\limits_{n\to\infty}(S_{2n}+u_{2n+1})$

or $\quad \lim\limits_{n\to\infty} S_{2n+1}=\lim\limits_{n\to\infty} S_{2n}+\lim\limits_{n\to\infty} u_{2n+1}=l+0,$ by (*ii*) and (*iii*) ...(*iv*)

$\therefore\quad \lim\limits_{n\to\infty} S_{2n+1}=l.$

From (*iii*), (*iv*), it follows that for any $\varepsilon > 0$, there exist positive integers $m_1$ and $m_2$ such that

$$|S_{2n} - l| < \varepsilon, \ \forall \quad n \geq m_1, \qquad ...(v)$$

and $$|S_{2n+1} - l| < \varepsilon, \ \forall \quad n \geq m_2. \qquad ...(vi)$$

Let $m = \max(m_1, m_2)$ so that $m \geq m_1, m \geq m_2$, ...(*vii*)

From (*v*), (*vi*), (*vii*) ; $|S_n - l| < \varepsilon \quad \forall \quad n \geq m$

$\Rightarrow \langle S_n \rangle$ converges to $l$.

Hence $\sum\limits_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent.

**Remark.** *The alternating series* $\Sigma(-1)^{n-1} u_n$ ***will not be convergent*** *if either* (*i*) $u_{n+1} \nless u_n, \forall n$ *or* $\lim\limits_{n \to \infty} u_n \neq 0$.

For example, the series $1 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + ...$ is not convergent,

$$\because \lim_{n \to \infty} u_n = \lim_{n \to \infty} \left(\frac{n+1}{n}\right) = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1 \neq 0.$$

**Example 77.** *Test for convergence the series*

(*a*) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$

(*b*) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + .....$ [D.U., Maths (H) 1999]

**Solution.** (*a*) We see that

$$\frac{1}{2} < 1, \frac{1}{3} < \frac{1}{2}, \frac{1}{4} < \frac{1}{3} \text{ and so on. Thus}$$

(*i*) $u_{n+1} < u_n \ \forall \ n$,

(*ii*) $\lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} \frac{1}{n} = 0.$

Hence, by Leibnitz's Test, the given series is convergent.

(*b*) The proof is similar. The given series is convergent.

**Example 78.** *Test for convergence* $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

**(Ans. Convergent)**

**Please try yourself.**

### 4.13 Absolute Convergence

*A series* $\Sigma u_n$ ***is said to be*** **absolutely convergent** *if the series* $\boldsymbol{\Sigma |u_n|}$ *is convergent.*

**Illustrations**

1. The series $\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + ...$

is absolutely convergent, since

$\Sigma |u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...$, being a geometric series with common ratio $\frac{1}{2} < 1$, is convergent.

2. The series $\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$

is not absolutely convergent, since

$\Sigma |u_n| = 1 + \frac{1}{2} + ... + \frac{1}{n} + ...$ is not convergent.

**Example 79.** *Discuss the absolute convergence and convergence of*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}. \qquad \text{[D.U., B.Sc. (G) 1993]}$$

**Hint.** Given series is convergent (See Example 77), but not absolutely convergent. (See Illustration 2 above)

### 4.14 Conditional Convergence

*A series* $\Sigma u_n$ *is said to be* **conditionally convergent,** *if*

(*i*) $\Sigma u_n$ *is convergent, and*

(*ii*) $\Sigma u_n$ *is not absolutely convergent.*

**Illustration**

The series $\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$

is conditionally convergent, since the given series is convergent (by Leibnitz Test) and

$$\Sigma |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... \text{ is not convergent,}$$

*i.e.,* $\Sigma u_n$ is not absolutely convergent.

**Ex.** *Define absolute convergence and conditional convergence of an infinite series. Give an example of a conditionally convergent series.*

[D.U., B.Sc. (G) 1996]

**Theorem 4.** *Show that every absolutely convergent series is convergent. The converse need not be true.* [D.U., Maths (H) 1999, 97, 95, 94 ; B.Sc. (G) 1997, 94 ; Physics (H) 1998, 93]

**Proof. Let** $\boldsymbol{\Sigma u_n}$ **be** an absolutely convergent series.

Then $\Sigma |u_n|$ is convergent.

By Cauchy's General Principle of Convergence (Theorem 2), for any $\varepsilon > 0$, there exists a positive integer $m$ such that

By Leibnitz Test, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$ converges.

Now $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges

Hence the given series is absolutely convergent.

**Example 81.** *Test for convergence and absolute convergence the series :*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots \text{for } p > 0.$$

[D.U., Physics (H) 1995 ; B.Sc. (G) 1999 ; Maths (H) 1988]

**Solution.** Since $p > 0$ and $(n+1)^p > n^p$, so

$$\frac{1}{(n+1)^p} < \frac{1}{n^p} \Rightarrow u_{n+1} < u_n \ \forall\ n.$$

Also $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n^p} = 0$, since $p > 0$.

Be Leibnitz's Test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ is convergent.

Now $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^p} \right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$. Hence the given series is absolutely convergent if $p > 1$ and conditionally convergent if $0 < p \le 1$.

**Example 82.** *Test for convergence and absolute convergence the series :*

$$\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} + \ldots\ldots$$

[D.U., Maths (H) 1995]

**Solution.** We have $u_{n+1} < u_n$ for each $n$ and

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{\sqrt{2n-1}} = 0.$$

Thus $\Sigma u_n$ is convergent, by Leibnitz's test. Now we show $\Sigma u_n$ is not absolutely convergent. We have

$$|u_n| = \frac{1}{\sqrt{2n-1}}. \text{ Let } v_n = \frac{1}{\sqrt{n}} \Rightarrow \lim_{n \to \infty} \frac{|u_n|}{v_n} = \lim_{n \to \infty} \sqrt{\frac{n}{2n-1}} = \frac{1}{\sqrt{2}} \neq 0.$$

Since $\Sigma\, v_n$ is divergent, so $\Sigma\, |u_n|$ is divergent.

Hence $\Sigma\, u_n$ is not absolutely convergent.

**Example 83.** *Show that for any fixed value of x, the series*

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} \textit{ is convergent.}$$

[D.U., Physics (H) 1996]

**Example 87.** *Discuss the convergence of the series :*

$$\sum_{n=1}^{\infty} \frac{\sin nx + \cos nx}{n^{3/2}}. \qquad \text{[D.U., Maths (H) 1996]}$$

**Solution.** We have

$$|u_n| = \left| \frac{\sin nx + \cos nx}{n^{3/2}} \right| \le \frac{1}{n^{3/2}} + \frac{1}{n^{3/2}} = \frac{2}{n^{3/2}}.$$

$$[\because |\sin nx| \le 1, \ |\cos nx| \le 1 \ \forall \ x]$$

Since $\Sigma \frac{1}{n^{3/2}}$ is convergent, so by First Comparison Test, $\Sigma |u_n|$ is convergent *i.e.*, $\Sigma u_n$ is absolutely convergent. Hence $\Sigma u_n$ is convergent.

**Example 88.** *Test for absolute convergence the series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(n+1)!}. \qquad \text{[D.U., Maths (H) 1996, 94 ; B.Sc. (G) 1996]}$$

**Solution.** We have

$$\lim_{n \to \infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n \to \infty} \frac{n^2}{(n+1)!} \times \frac{(n+2)!}{(n+1)^2} = \lim_{n \to \infty} \frac{(n+2)}{(1+1/n)^2} = \infty.$$

By Ratio Test, $\Sigma |u_n|$ is convergent. Hence the given series is absolutely convergent.

**Example 89.** *Show that the series :*

$$\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \frac{1}{7.8} + \text{.... } \textit{is absolutely convergent.}$$

[D.U., Physics (H) 1994]

**Solution.** We have

$$|u_n| = \frac{1}{(2n-1)2n}. \text{ Let } v_n = \frac{1}{n^2}.$$

$$\therefore \quad \frac{|u_n|}{v_n} = \frac{n^2}{2n(2n-1)} = \frac{n}{2(2n-1)} = \frac{1}{4} \quad \text{as } n \to \infty.$$

Thus $\Sigma |u_n|$ and $\Sigma v_n$ converge or diverge together.

Since $\Sigma v_n = \Sigma 1/n^2$ is convergent, so $\Sigma |u_n|$ is convergent.

Hence $\Sigma u_n$ is absolutely convergent.

**Example 90.** *Test for convergence the series :*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\sqrt{n+1} - \sqrt{n-1}). \qquad \text{[D.U., Physics (H) 1995]}$$

**Solution.** We have

$$|u_n| = \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$$

**Solution.** $|u_n| = \dfrac{1}{\sqrt{n^5}} + \dfrac{1}{\sqrt{(n+1)^5}} < \dfrac{1}{\sqrt{n^5}} + \dfrac{1}{\sqrt{n^5}}\ \forall\ n$

*i.e.,* $$|u_n| < 2\left(\frac{1}{n^{5/2}}\right) \forall\ n.$$

Since $\Sigma \dfrac{1}{n^{5/2}}$ is convergent, so by First Comparison Test, $\Sigma |u_n|$ is convergent. Hence $\Sigma\, u_n$ is absolutely convergent and so convergent.

**Example 94.** *Test the conditional and absolute convergence of*

$$(i)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}, \qquad (ii)\ \sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{n^3} + \frac{1}{(n+1)^3}\right].$$

[D.U. Physics (H) 1996]

**Solution.** (*i*) We have $u_{n+1} < u_n$ for each $n$ and $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0.$

By Leibnitz's test, $\Sigma\, u_n$ is convergent.

Now $\Sigma |u_n| = \Sigma \dfrac{1}{\sqrt{n}}$ is not convergent $\quad (\because p = \frac{1}{2} < 1).$

Hence $\Sigma\, u_n$ is not absolutely convergent but it is conditionally convergent.

(*ii*) We have

$$|u_n| = \frac{1}{n^3} + \frac{1}{(n+1)^3} < \frac{1}{n^3} + \frac{1}{n^3} = \frac{2}{n^3}\ \forall\ n.$$

By First Comparison Test, $\Sigma |u_n|$ is convergent ($\because \Sigma\, 1/n^3$ is convergent). Hence $\Sigma\, u_n$ is absolutely convergent. Since $\Sigma\, u_n$ is absolutely convergent, so by definition, it cannot be conditionally convergent.

**Example 95.** *Test for convergence the series*

$$\left(\frac{1}{3}\right)^2 - \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 - \dots$$

**Solution.** The series of absolute terms

$$\Sigma |u_n| = \left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \dots$$

is convergent (See Example 44).

Hence the given series is absolutely convergent, and so convergent.

**Example 96.** *Test for convergence the series*

$$\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{2^n+5}.$$

**Solution.** We have

$$|u_n| = \frac{n+2}{2^n+5},\ |u_{n+1}| = \frac{n+3}{2^{n+1}+5}.$$

$$\therefore \quad \frac{|u_n|}{|u_{n+1}|} = \left(\frac{n+2}{n+3}\right)\left(\frac{2^{n+1}+5}{2^n+5}\right),$$

and $$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \left(\frac{1+2/n}{1+3/n}\right)\left(\frac{2+5/2^n}{1+5/2^n}\right) = 2 > 1.$$

By Ratio Test, $|u_n|$ converges. Hence $\Sigma u_n$ is absolutely convergent and so convergent.

**Example 97.** *Test for convergence and absolute convergence the series* :

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)} = \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \dots$$

[D.U., Maths (H) 1998]

**Solution.** We have $u_n = \dfrac{1}{\log(n+1)}$ $\forall\, n \in \mathbf{N}$.

Clearly, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\log(n+1)} = 0.$$

Since $\log x$ is an increasing function for all $x > 0$, therefore

$$\log(n+2) > \log(n+1) \qquad (\because\ n+2 > n+1)$$

$$\Rightarrow \quad \frac{1}{\log(n+2)} < \frac{1}{\log(n+1)}\ \forall\ n \Rightarrow u_{n+1} < u_n\ \forall\ n.$$

Thus the two conditions of Leibnitz Test are satisfied and so the given series is convergent.

Now we test the absolute convergence of the given series.

We have $$\sum_{n=2}^{\infty} |u_n| = \sum_{n=2}^{\infty} \frac{1}{\log n},$$ which is divergent.

[See Example 16]

Hence $\Sigma u_n$ is not absolutely convergent *i.e.*, $\Sigma u_n$ is conditionally convergent.

**Example 98.** *Test the convergence and absolute convergence of the series*

$$\frac{1}{2\log 2} - \frac{1}{3\log 3} + \frac{1}{4\log 4} - \dots$$

**Solution.** We have $u_n = \dfrac{1}{(n+1)\log(n+1)}$ $\forall\, n \in \mathbf{N}$.

Clearly $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(n+1)\log(n+1)} = 0.$$

Let $u(x) = \dfrac{1}{x\log x}$. Then $u'(x) = \dfrac{-(1+\log x)}{(x\log x)^2}$.

$\Rightarrow u'(x) < 0\ \forall\ x > 0$

$\Rightarrow u(x)$ is a decreasing function $\forall\ x > 0$

$\Rightarrow u(n+2) \le u(n+1)\ \forall\ n \qquad (\because\ n+2 > n+1)$

$\Rightarrow u_{n+1} \le u_n\ \forall\ n.$

Thus both the conditions of Leibnitz test are satisfied and so the given series is convergent.

Now $$\Sigma|u_n| = \sum_{n=2}^{\infty} \frac{1}{n\log n}$$ is divergent. [See Example 74]

Hence $\Sigma u_n$ is not absolutely convergent.

**Example 99.** *Show that the series*

$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \dots \textit{ converges.}$$

[D.U., B.A. (P) 1996]

**Solution.** We have $u_n = \dfrac{\log(n+1)}{(n+1)^2}$ $\forall\, n \in \mathbf{N}$.

Since $$\lim_{n\to\infty} \frac{\log n}{n^2} = 0,\ \lim_{n\to\infty} u_n = 0.$$

Now we shall prove that $u_{n+1} \le u_n\ \forall\ n$.

Let $f(x) = \dfrac{\log x}{x^2}$ so that $f'(x) = \dfrac{x^2(1/x) - 2x\log x}{x^4}$

$$\Rightarrow \quad f'(x) = \frac{1-2\log x}{x^3} < 0\ \ \forall\ \ x > e^{1/2}$$

$$(\because\ x > e^{1/2} \Leftrightarrow \log x > \tfrac{1}{2} \Leftrightarrow 1 - 2\log x < 0)$$

$\Rightarrow \quad f(x)$ is a decreasing function $\forall\ x > e^{1/2}$

$\Rightarrow \quad f(n+2) \le f(n+1)\ \ \forall\ n \qquad (\because\ n+2 > n+1 > e^{1/2}\ \forall\ n)$

$$\Rightarrow \quad \frac{\log(n+2)}{(n+2)^2} \le \frac{\log(n+1)}{(n+1)^2}\ \ \forall\ n$$

$\Rightarrow \quad u_{n+1} \le u_n\ \ \forall\ n.$

Thus both the conditions of Leibnitz test are satisfied and so the given series is convergent.

**Example 100.** *Test for convergence and absolute convergence* :

$$\frac{1}{2(\log 2)^p} - \frac{1}{3(\log 3)^p} + \frac{1}{4(\log 4)^p} - \dots\dots (p > 0).$$

**Solution.** We have

$$\sum_{n=2}^{\infty} |u_n| = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p},$$

which is convergent if $p > 1$ and divergent if $0 < p \le 1$ [See Example 73].

Hence the given series is absolutely convergent if $p > 1$.

We now consider the case $0 < p \le 1$. We have

$$u_n = \frac{1}{(n+1)\{\log(n+1)\}^p} \quad \forall\, n \in \mathbf{N}.$$

Then $\lim\limits_{n\to\infty} u_n = 0$. Since $\log x$ is an increasing function for $x > 0$, so

$$\log(n+2) > \log(n+1) \;\Rightarrow\; \{\log(n+2)\}^p > \{\log(n+1)\}^p \qquad (\because 0 < p \le 1)$$

$\Rightarrow (n+2)\{\log(n+2)\}^p > (n+1)\{\log(n+1)\}^p \Rightarrow u_{n+1} < u_n \;\forall\, n.$

By Leibnitz's Test, $\Sigma\, u_n$ is convergent for $0 < p \le 1$.

As shown above, $\Sigma\, u_n$ is not absolutely convergent for $0 < p \le 1$.

Hence $\Sigma\, u_n$ is conditionally convergent for $0 < p \le 1$.

**Example 101.** *Show that the series*

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

*converges absolutely for all values of x.* [D.U., B.A. (P) 1991]

**Solution.** We have $u_n = \dfrac{x^n}{n!},\; u_{n+1} = \dfrac{x^{n+1}}{(n+1)!}$.

Now $\lim\limits_{n\to\infty} \dfrac{|u_n|}{|u_{n+1}|} = \lim\limits_{n\to\infty} \dfrac{(n+1)!}{n!\,.\,|x|} = \lim\limits_{n\to\infty} \dfrac{n+1}{|x|} + \infty$, provided $x \ne 0$.

By Ratio test, $\Sigma\,|u_n|$ is convergent for all values of $x \ne 0$. But for $x = 0$, the given series is obviously convergent.

Hence the given series converges absolutely for all values of $x$.

**Example 102.** *Show that the series* $1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots$ *converges absolutely for all values of x.* [D.U., Maths (H) 1994]

**Solution.** On ignoring the first term, this problem is Example 101.

**Example 103.** *Show that the series*

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \text{ is absolutely convergent.}$$

**Solution.** Ignoring the first term, we have

$$|u_n| = \frac{x^{2n}}{(2n)!},\; |u_{n+1}| = \frac{x^{2n+2}}{(2n+2)!}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!}\cdot\frac{1}{x^2}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{x^2} = +\infty, \text{ provided } x \ne 0.$$

By Ratio test, $\Sigma\,|u_n|$ is convergent $\forall\, x \ne 0$.

But for $x = 0$, the given series is obviously convergent. Hence $\Sigma\, u_n$ is absolutely convergent for all values of $x$. In other words, $\Sigma\, u_n$ is convergent for all values of $x$.

**Example 104.** *Prove that the series*

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

*converges if and only if* $-1 \le x \le 1$. [D. U., B.Sc. (G) 1995, 90]

**Solution.** Here $|u_n| = \left|\dfrac{x^{2n-1}}{2n-1}\right|$ and $|u_{n+1}| = \left|\dfrac{x^{2n+1}}{2n+1}\right|$

$$\therefore \quad \lim_{n\to\infty} \left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{x^2}\cdot\frac{2n+1}{2n-1} = \frac{1}{x^2}.$$

By Ratio test, $\Sigma\,|u_n|$ is convergent, if

$$\frac{1}{x^2} > 1 \;\Rightarrow\; x^2 < 1 \;\Rightarrow\; |x| < 1,$$

and $\quad \Sigma\,|u_n|$ is divergent if $|x| > 1$.

Thus $\quad \Sigma\, u_n$ is absolutely convergent if $|x| < 1$

$\Rightarrow \quad \Sigma\, u_n$ is convergent if $|x| < 1$.

If $\quad |x| = 1$, the Ratio test fails.

Now $\quad |x| = 1 \;\Rightarrow x = 1$ or $x = -1$.

Putting $x = 1$, the given series becomes

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots$, which converges by Leibnitz's Test.

Putting $x = -1$, the series becomes $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} \dots$ which is again convergent by Leibnitz's Test.

Hence the given series is convergent if $|x| \le 1$ or $-1 \le x \le 1$.

If $x > 1$, then $u_n$ does not tend to zero as $n \to \infty$, and so $\Sigma\, u_n$ does not converge when $x > 1$.

Since $\Sigma\,|u_n|$ diverges when $x < -1$ (*i.e.*, $|x| > 1$), so $\Sigma\, u_n$ also diverges when $x < -1$.

Hence $\Sigma\, u_n$ converges iff $-1 \le x \le 1$.

**Example 105.** *Show that the series*

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

*is absolutely convergent if and only if* $|x| < 1$ *but conditionally convergent for* $|x| = 1$. [D.U., Physics (H) 1993]

**9.** Test for convergence and absolute convergence of the series

$$\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}-\frac{1}{7.8}+\ldots \qquad \textbf{(Ans. Absolutely convergent)}$$

$$\left[\textbf{Hint. } |u_n| = \frac{1}{(2n-1)\,2n} \simeq \frac{1}{n^2}.\right]$$

**10.** Show that the series : $1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\ldots$ is absolutely convergent.

**11.** Show that : $1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\ldots.$ is absolutely convergent.

[**Hint.** $|u_n| = \frac{1}{(2n)!}, |u_{n+1}| = \frac{1}{2(n+1)!}$.

$$\therefore \quad \frac{|u_n|}{|u_{n+1}|} = \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \to \infty \text{ as } n \to \infty.$$

By Ratio Test, $\Sigma\,|u_n|$ is convergent *i.e.*, $\Sigma\, u_n$ is absolutely convergent.]

**12.** Test the absolute convergence of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1}\left(\sqrt{n^2+1}-n\right).$$

[**Hint.** Similar to Example 90.] [**Ans.** Not Absolutely Convergent]

**13.** Test for convergence the series

$$1-\frac{(2\,!)^2}{4\,!}+\frac{(3\,!)^2}{6\,!}-\ldots \qquad \textbf{(Ans. Convergent)}$$

[**Hint.** Prove that the series is absolutely convergent by applying Ratio Test.]

**14.** Test the convergence and absolute convergence of the series

$$1-\frac{1}{4.3}+\frac{1}{4^2.5}-\frac{1}{.\,4^3.7}+\ldots$$

[**Hint.** $|u_n| = \frac{1}{(2n-1)\,4^{n-1}}, |u_{n+1}| = \frac{1}{(2n+1)\,4^n}$.

$\therefore \lim\limits_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = 4 > 1$. Hence $\Sigma\,|u_n|$ is convergent.]

**15.** Show that the series : $\frac{1}{2-\sqrt{2}}-\frac{1}{3-\sqrt{3}}+\frac{1}{4-\sqrt{4}}-\ldots$

is conditionally convergent.

[**Hint.** Given series is convergent, by Leibnitz Test.

Notice that $\lim\limits_{n\to\infty} \frac{1}{n-\sqrt{n}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}\,(\sqrt{n}-1)} = 0.$

If the given series is absolutely convergent, then $\sum^{\infty} \frac{1}{\ }$ is convergent.

**Example 3.** *Show that the function f, defined on* $\mathbf{R}-\{0\}$ *by* $f(x)=\sin(1/x)$, *whenever* $x\neq 0$, *does not approach* 0 *as* $x\to 0$.

**Solution.** Let $\varepsilon=\frac{1}{2}>0$ and let $\delta$ be any positive number.

By Archimedean property, for $1/\delta>0$, $\exists\, n\in\mathbf{N}$ such that

$$2n\pi+\frac{\pi}{2}>\frac{1}{\delta} \quad \text{or} \quad \frac{1}{2n\pi+\pi/2}<\delta.$$

If $x=\dfrac{1}{2n\pi+\pi/2}$, then $0<|x-0|<\delta$.

Now $|\sin(1/x)-0|=|\sin(2n\pi+\pi/2)=1>\varepsilon$.

Thus there is an $\varepsilon=\frac{1}{2}>0$, such that for each $\delta>0$, $\exists$ some point

$x=\dfrac{1}{2n\pi+\pi/2}$, for which $|\sin(1/x)-0|>\varepsilon$ and $0<|x-0|<\delta$.

Hence $\lim\limits_{x\to}\sin(1/x)\neq 0$.

## 5.2 Some Theorems on Limits

**Theorem 1.** *$\lim\limits_{x\to a} f(x)$, if it exists, is unique.*

**Proof.** Let $\lim\limits_{x\to a} f(x)=l$ and $\lim\limits_{x\to a} f(x)=l'$. ...(1)

We shall show that $l=l'$. Let $\varepsilon>0$ be given.

Using (1), there exists $\delta_1>0$ and $\delta_2>0$ such that

$$|f(x)-l|<\frac{1}{2}\varepsilon, \quad \text{when } 0<|x-a|<\delta_1, \qquad ...(2)$$

and

$$|f(x)-l'|<\frac{1}{2}\varepsilon, \quad \text{when } 0<|x-a|<\delta_2. \qquad ...(3)$$

Let $\delta=\min(\delta_1,\delta_2)$. Then, by (2) and (3),

$$|f(x)-l|<\frac{1}{2}\varepsilon, \quad \text{when } |x-a|<\delta,$$

$$|f(x)-l'|<\frac{1}{2}\varepsilon, \quad \text{when } |x-a|<\delta.$$

Now

$$\begin{aligned}|l-l'|&=|l-f(x)+f(x)-l'|\\ &\leq|l-f(x)|+|f(x)-l'|<\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon=\varepsilon\end{aligned}$$

$\therefore\ |l-l'|<\varepsilon$, when $0<|x-a|<\delta$.

Since $\varepsilon$ is arbitrarily small, $|l-l'|=0$. Hence $l=l'$.

**Definition.** *Let f and g be two real-valued functions with a common domain D i.e., $f, g: D\to\mathbf{R}$. We define*

**Sum** $(f+g)(x)=f(x)+g(x) \quad \forall\ x\in D$

**Difference** $(f-g)(x)=f(x)-g(x), \quad \forall\ x\in D$

Let us choose $\varepsilon'$ such that

$$\varepsilon' < \frac{\varepsilon}{(|m|+|l|+1)}, \varepsilon > 0.$$

$\therefore \quad |f(x)\,g(x) - lm| < \varepsilon,$ when $0 < |x-a| < \delta.$

Hence $\lim\limits_{x \to a} [f(x)\,g(x)] = lm = \lim\limits_{x \to a} f(x) \,.\, \lim\limits_{x \to a} g(x).$

**Theorem 4.** *If* $\lim\limits_{x \to a} f(x) = l \neq 0$*, then there exists a* $\delta > 0$ *such that*

$$|f(x)| > \frac{1}{2}|l|, \textit{ when } 0 < |x-a| < \delta.$$

**Proof.** Since $l \neq 0$, so $|l| > 0$.

Let us choose $\varepsilon > 0$ such that $\varepsilon = \frac{1}{2}|l| > 0.$

Now $\lim\limits_{x \to a} f(x) = l \Rightarrow$ for any $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$|f(x) - l| < \varepsilon, \text{ when } 0 < |x-a| < \delta. \quad \text{...(1)}$$

Now $\quad |l| = |l - f(x) + f(x)| \leq |l - f(x)| + |f(x)|. \quad$ ...(2)

From (1) and (2), $|l| < \varepsilon + |f(x)|$, when $0 < |x-a| < \delta$

$\Rightarrow |f(x)| > |l| - \varepsilon,$ when $0 < |x-a| < \delta$

$\Rightarrow |f(x)| > |l| - \dfrac{|l|}{2} = \dfrac{|l|}{2},$ when $0 < |x-a| < \delta$

Hence $|f(x)| > \frac{1}{2}|l|,$ when $0 < |x-a| < \delta.$

**Theorem 5.** *If* $\lim\limits_{x \to a} f(x) = l$ *and* $\lim\limits_{x \to a} g(x) = m$*, and* $m \neq 0$*, then*

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{l}{m}. \qquad \text{[D.U., B.Sc. (G) 1993]}$$

**Proof.** Consider $\left|\dfrac{f(x)}{g(x)} - \dfrac{l}{m}\right| = \dfrac{|mf(x) - lg(x)|}{|g(x)|\,.\,|m|}$

$$= \frac{|mf(x) - lm + lm - lg(x)|}{|g(x)|\,.\,|m|}$$

$$= \frac{|m(f(x) - l) + l(m - g(x))|}{|g(x)|\,.\,|m|}$$

$$\leq \frac{|m|\,.\,|f(x) - l| + |l|\,|g(x) - m|}{|m|\,.\,|g(x)|} \quad \text{...(1)}$$

Since $\lim\limits_{x \to a} g(x) = m \neq 0$, so there exists a $\delta_1 > 0$ such that

$$|g(x)| > \frac{1}{2}|m|, \text{ when } 0 < |x-a| < \delta_1 \qquad \text{(Theorem 4)}$$

$$\Rightarrow \quad \frac{1}{|g(x)|} < \frac{2}{|m|}, \text{ when } 0 < |x-a| < \delta_1. \quad \text{...(2)}$$

**2.** Clearly, *$\lim\limits_{x \to a} f(x)$ exists if* ***and only if*** *$\lim\limits_{x \to a-} f(x)$ and $\lim\limits_{x \to a+} f(x)$ both exist and are equal. Hence*

$$\lim_{x \to a} f(x) = \lim_{x \to a-} f(x) = \lim_{x \to a+} f(x).$$

**Theorem 6.** *Prove that $\lim\limits_{x \to c} f(x)$ exists and is equal to a number $l$ if and only if both left limit $\lim\limits_{x \to c-0} f(x)$ and right limit $\lim\limits_{x \to c+0} f(x)$ exist and are equal to $l$.* [D.U., B.Sc. (G) 1996]

*Or*

*Let $f$ be defined on a deleted neighbourhood of $c$. Show that $\lim\limits_{x \to c} f(x)$ exists and equals $l$ iff $f(c+0)$, $f(c-0)$ both exist and are equal to $l$.*

**Proof.** *Condition is necessary*

Let $\lim\limits_{x \to c} f(x) = l$. Then for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$|f(x) - l| < \varepsilon, \quad \text{when} \quad 0 < |x - c| < \delta$$

or $$|f(x) - l| < \varepsilon, \quad \text{when} \quad c - \delta < x < c + \delta, \ x \neq c.$$

It follows that $$|f(x) - l| < \varepsilon, \quad \text{when} \quad c - \delta < x < c, \qquad \text{...(1)}$$

and $$|f(x) - l| < \varepsilon, \quad \text{when} \quad c < x < c + \delta. \qquad \text{...(2)}$$

From (1) and (2), we see that

$$\lim_{x \to c-0} f(x) \ \text{ and } \ \lim_{x \to c+0} f(x) \text{ both exist and are equal to } l.$$

*Condition is sufficient*

Let $$\lim_{x \to c+0} f(x) = l = \lim_{x \to c-0} f(x).$$

Then for any $\varepsilon > 0$, there exist some $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - l| < \varepsilon, \quad \text{when} \quad c < x < c + \delta_1, \qquad \text{...(3)}$$

and $$|f(x) - l| < \varepsilon, \quad \text{when} \quad c - \delta_2 < x < c. \qquad \text{...(4)}$$

Let $\delta = \min(\delta_1, \delta_2)$. Then $\delta \leq \delta_1$ and $\delta \leq \delta_2$

$$\Rightarrow \quad c + \delta \leq c + \delta_1 \quad \text{and} \quad c - \delta \geq c - \delta_2 \ \ (\text{or } c - \delta_2 \leq c - \delta). \qquad \text{...(5)}$$

From (3), and (5), we obtain

$$|f(x) - l| < \varepsilon, \quad \text{when} \quad c < x < c + \delta,$$

and $$|f(x) - l| < \varepsilon, \quad \text{when} \quad c - \delta < x < c.$$

$\therefore$ $$|f(x) - l| < \varepsilon, \quad \text{when} \quad c - \delta < x < c + \delta, \ x \neq c$$

or $$|f(x) - l| < \varepsilon, \quad \text{when} \quad 0 < |x - c| < \delta.$$

Hence $\lim\limits_{x \to c} f(x) = l$.

## EXAMPLES

**Example 4.** *Evaluate* $\lim\limits_{x \to 0-} \dfrac{|x|}{x}$ *and* $\lim\limits_{x \to 0+} \dfrac{|x|}{x}$.

## 5.4 Infinite Limits and Limits at Infinity

**Definition 1.** *The expression* $\lim_{x \to +\infty} f(x) = l$ *means,*

*given any* $\varepsilon > 0$, *there exists some* $M > 0$ *such that*

$$|f(x) - l| < \varepsilon, \text{ when } x > M.$$

**Definition 2.** *The expression* $\lim_{x \to -\infty} f(x) = l$ *means,*

*given any* $\varepsilon > 0$, *there exists some* $M > 0$ *such that*

$$|f(x) - l| < \varepsilon, \text{ when } x < -M.$$

**Definition 3.** *The expression* $\lim_{x \to a} f(x) = +\infty$ *means,*

*given any* $M > 0$ *(however large), there exists some* $\delta > 0$ *such that*

$$f(x) > M, \text{ when } |x - a| < \delta.$$

**Definition 4.** *The expression* $\lim_{x \to a} f(x) = -\infty$ *means,*

*given any* $M > 0$ *(however large), there exists some* $\delta > 0$ *such that*

$$f(x) < -M, \text{ when } |x - a| < \delta.$$

### EXAMPLES

**Example 8.** *Show that* $\lim_{x \to +\infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0.$

**Solution.** Let $\varepsilon > 0$ be given. Choose $M = 1/\varepsilon$. Then

$$x > M \Rightarrow \left|\frac{1}{x}\right| < \frac{1}{M} = \varepsilon \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon.$$

Hence $\lim_{x \to +\infty} \frac{1}{x} = 0$, by Definition 1.

Now $x < -M \Rightarrow |x| > M \Rightarrow \left|\frac{1}{x}\right| < \frac{1}{M} = \varepsilon$

or $x < -M \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$. Hence $\lim_{x \to -\infty} \frac{1}{x} = 0.$

**Example 9.** *Show that*

(*i*) $\lim_{x \to 0+} \frac{1}{x} = +\infty$, (*ii*) $\lim_{x \to 0-} \frac{1}{x} = -\infty$, (*iii*) $\lim_{x \to 0} \frac{1}{x^2} = +\infty.$

**Solution.** (*i*) Let $M > 0$ be any large number. Let $\delta = 1/M$.

Then $0 < x < \delta \Rightarrow \frac{1}{x} > \frac{1}{\delta} \Rightarrow \frac{1}{x} > M.$

Hence $\lim_{x \to 0+} \frac{1}{x} = +\infty$, by Definition 3.

and $$\lim_{x\to 0+} \frac{e^{1/x}-1}{e^{1/x}+1} = \lim_{x\to 0+} \frac{1-e^{-1/x}}{1+e^{-1/x}} = \frac{1-0}{1+0} = 1, \text{ by (A).}$$

Hence $\lim\limits_{x\to 0} \dfrac{e^{1/x}-1}{e^{1/x}+1}$ does not exist. $(\because -1 \neq 1)$

**Example 12.** *Show that* $\mathit{lim}_{x\to 0} \dfrac{xe^{1/x}}{1+e^{1/x}} = 0.$

**Solution.** We know that $x \to 0+ \Rightarrow e^{-1/x} \to 0,$

and $x \to 0- \Rightarrow e^{1/x} \to 0.$ (See Example 11)

We have $$\lim_{x\to 0-} \frac{xe^{1/x}}{1+e^{1/x}} = \frac{0.0}{1+0} = 0,$$

and $$\lim_{x\to 0+} \frac{xe^{1/x}}{1+e^{1/x}} = \lim_{x\to 0+} \frac{x}{e^{-1/x}+1} = \frac{0}{0+1} = 0.$$

Hence $$\lim_{x\to 0} \frac{xe^{1/x}}{1+e^{1/x}} = 0.$$

## 5.5 Continuous Functions

Consider a function $f : [a, b] \to \mathbf{R}$. Let $a < c < b$.

**Definition 1.** *A function f is said to be* **continuous** *at a point c, if*

$$\mathit{lim}_{x\to c} f(x) = f(c).$$

*In other words, a function f is said to be* **continuous** *at a point c, if for any* $\varepsilon > 0$, *there exists some* $\delta > 0$ *such that*

$$|f(x) - f(c)| < \varepsilon, \quad \textit{when} \quad |x - c| < \delta. \qquad \ldots(1)$$

*Equivalently, a function f is said to be* **continuous** *at a point c, if*

$$\boxed{\mathit{lim}_{x\to c-} f(x) = \mathit{lim}_{x\to c+} f(x) = f(c).}$$

**Definition 2.** *A function f is said to be* **continuous** *at the left end point a, if*

$$\mathit{lim}_{x\to a+} f(x) = f(a).$$

**Definition 3.** *A function f is said to be* **continuous** *at the right end point b, if*

$$\mathit{lim}_{x\to b-} f(x) = f(b).$$

**Definition 4.** *A function f is said to be* **continuous** *in an interval [a, b], if it is continuous at every point of the interval [a, b].*

**Remark 1.** *If a function f* **is not continuous at a point** *c, then from (1), it follows that there exists* **some** $\varepsilon > 0$ *such that* **for each** $\delta > 0$, *there* **is some** *y satisfying*

$$|f(y) - f(c)| \geq \varepsilon, \textit{ when } |y - c| < \delta.$$

$$\lim_{x \to a+} f(x) = \lim_{x \to a} \frac{(x-a)}{(x-a)} = 1, \text{ as } x \neq a.$$

Thus $\lim_{x \to a} f(x)$ does not exist and so $f$ is discontinuous at $x = a$.

**Example 16.** *Examine the continuity at $x = 1$ and $x = 2$ of the function*

$$\begin{aligned} f(x) &= 2x && \text{if} \quad 0 \leq x \leq 1, \\ &= 2 - x && \text{if} \quad 1 < x \leq 2, \\ &= x^2 - 2x && \text{if} \quad x > 2. \end{aligned}$$

**Solution.** We have $\lim_{x \to 1-} f(x) = \lim_{x \to 1} 2x = 2,$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1} (2 - x) = 1.$$

Thus $\lim_{x \to 1} f(x)$ does not exist and so $f$ is discontinuous at $x = 1$.

Now
$$\lim_{x \to 2-} f(x) = \lim_{x \to 2} (2 - x) = 0,$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2} (x^2 - 2x) = 0.$$

Also
$$f(2) = 2 - 2 = 0.$$

Thus $\lim_{x \to 2-} f(x) = \lim_{x \to 2+} f(x) = f(2)$ and so $f$ is continuous at $x = 2$.

**Example 17.** *Show that the function*

$$\begin{aligned} f(x) &= \frac{x e^{1/x}}{1 + e^{1/x}}, && x \neq 0 \\ &= 0 && x = 0 \end{aligned}$$

*s continuous at $x = 0$.*

**Solution.** We have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x e^{1/x}}{1 + e^{1/x}} = 0. \qquad \text{(See Example 12)}$$

$$\therefore \quad \lim_{x \to 0} f(x) = f(0). \qquad (\because f(0) = 0)$$

Hence $f$ is continuous at $x = 0$.

**Example 18.** *Show that the function*

$$\begin{aligned} f(x) &= \frac{e^{1/x} - 1}{e^{1/x} + 1}, && x \neq 0 \\ &= 0, && x = 0 \end{aligned}$$

*is discontinuous at $x = 0$.*

**Solution.** Since $\lim_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ does not exist (Example 11), so $f$ is discontinuous at $x = 0$.

**Example 19.** *Show that the function*

$$f(x) = \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}, \quad x \neq 0$$
$$= 1, \quad x = 0$$

*is discontinuous at* $x = 0$.

**Solution.** We know $x \to 0- \;\Rightarrow\; e^{1/x} \to 0,$ ...(1)

and $x \to 0+ \;\Rightarrow\; e^{-1/x} \to 0.$ ...(2)

$$\therefore \quad \lim_{x \to 0-} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = \lim_{x \to 0-} \frac{e^{1/x} . e^{1/x} - 1}{e^{1/x} . e^{1/x} + 1} = \frac{0.0 - 1}{0.0 + 1} = -1, \text{ by (1)}$$

$$\lim_{x \to 0+} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = \lim_{x \to 0+} \frac{1 - e^{-1/x} . e^{-1/x}}{1 + e^{-1/x} . e^{-1/x}} = \frac{1 - 0.0}{1 + 0.0} = 1, \text{ by (2).}$$

Hence $\lim_{x \to 0} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ does not exist $\quad (\because -1 \neq 1)$

and so the given function is discontinuous at $x = 0$.

**Example 20.** *Show that the function f defined on* **R** *as*

$$f(x) = x \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}, x \neq 0$$

*and* $f(0) = 0$, *is continuous at* $x = 0$.

**Hint.** $\lim_{x \to 0-} f(x) = 0(-1) = 0$, by Example 19.

$\lim_{x \to 0+} f(x) = 0(+1) = 0.$ Also $f(0) = 0$.

**Example 21.** *Examine the continuity of the function* :

$$f(x) = \begin{cases} \dfrac{e^{1/x^2}}{1 - e^{1/x^2}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

*at* $x = 0$. [D.U., Maths (H) 1998]

**Solution.** Clearly, $x \to 0+$ and $x \to 0- \Rightarrow 1/x^2 \to +\infty \Rightarrow e^{1/x^2} \to +\infty$

$$\Rightarrow 1/e^{1/x^2} \to 0 \Rightarrow e^{-1/x^2} \to 0.$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{e^{1/x^2}}{1 - e^{1/x^2}} = \lim_{x \to 0-} \frac{1}{e^{-1/x^2} - 1} = \frac{1}{0 - 1} = -1.$$

Similarly, $\lim_{x \to 0+} f(x) = -1$ and so $\lim_{x \to 0} f(x) = -1 \neq f(0) \quad [\because f(0) = 0]$

Hence $f$ is discontinuous at $x = 0$.

**Example 22.** *Show that the function*

$$f(x) = \begin{cases} \dfrac{e^{1/x^2}}{1 - e^{1/x^2}}, & \text{when } x \neq 0 \\ -1, & \text{when } x = 0 \end{cases}$$

*is continuous at* $x = 0$.

**Hint.** See Example 21. Here $f(0) = -1$.

**Example 23.** *Show that the function f defined or* **R** *by setting*

$$f(x) = \frac{x}{1 + e^{1/x}}, \text{ when } x \neq 0 \text{ and } f(0) = 0$$

*is continuous at* $x = 0$.

**Solution.** $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x}{1 + e^{1/x}} = \frac{0}{1 + 0} = 0, \quad \left( \because \lim_{x \to 0-} e^{1/x} = 0 \right)$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x}{1 + e^{1/x}} = \lim_{x \to 0+} \frac{x e^{-1/x}}{e^{-1/x} + 1} = \frac{0.0}{0 + 1} = 0. \quad \left( \because \lim_{x \to 0+} e^{-1/x} = 0 \right)$$

Also $f(0) = 0$. Hence $f$ is continuous at $x = 0$.

**Example 24.** *Show that*

$$f(x) = |x| + |x - 1|$$

*is continuous at* $x = 0$ *and* $x = 1$.

**Solution.** $f(x) = -x - (x - 1) = 1 - 2x$, when $x < 0$ ; ...(1)

$f(x) = x - (x - 1) = 1$ when $0 \leq x < 1$ ; ...(2)

$f(x) = x + (x - 1) = 2x - 1$, when $x \geq 1$. ...(3)

Now $\lim_{x \to 0-} f(x) = \lim_{x \to 0} (1 - 2x) = 1$, by (1)

and $\lim_{x \to 0+} f(x) = \lim_{x \to 0} (1) = 1$, by (2).

Also $f(0) = |0| + |0 - 1| = 1$.

$\therefore \quad \lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0).$

Hence $f$ is continuous at $x = 0$.

Now $\lim_{x \to 1-} f(x) = \lim_{x \to 1} (1) = 1$, by (2)

$\lim_{x \to 1+} f(x) = \lim_{x \to 1} (2x - 1) = 1$, by (3).

Also $f(1) = |1| + |1 - 1| = 1$.

$\therefore \quad \lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(1).$

Hence $f$ is continuous at $x = 1$.

**Example 25.** *Show that*

$$f(x) = |x-1| + |x-2|$$

*is continuous at* $x = 1$ *and* $x = 2$.

**Hint.** $f(x) = (1-x) + (2-x)$, if $x < 1$
$= (x-1) + (2-x)$, if $1 \le x < 2$
$= (x-1) + (x-2)$, if $x \ge 2$.

**Example 26.** *Let f be the function defined on* **R** *by setting*

$$f(x) = [x], \text{ for all } x \in \mathbf{R},$$

*where* $[x]$ *denotes the greatest integer not exceeding x. Show that f is discontinuous at the points* $x = 0, \pm 1, \pm 2, \pm 3, \ldots$ *and is continuous at every other point.* [D.U., B.Sc. (G) 1991]

**Solution.** By definition, we have

$$\begin{aligned} [x] &= 0, \text{ for } 0 \le x < 1, \\ [x] &= 1, \text{ for } 1 \le x < 2, \\ [x] &= 2, \text{ for } 2 \le x < 3, \\ &\vdots \\ [x] &= -1, \text{ for } -1 \le x < 0. \\ [x] &= -2 \text{ for } -2 \le x < -1 \text{ and so on.} \end{aligned}$$

**At $x = 0$**

$$\lim_{x \to 0-} f(x) = \lim_{\substack{x \to 0 \\ (x<0)}} [x] = -1,$$

$$\lim_{x \to 0+} f(x) = \lim_{\substack{x \to 0 \\ (x>0)}} [x] = 0.$$

Since $\lim\limits_{x \to 0-} f(x) \ne \lim\limits_{x \to 0+} f(x)$, $f$ is discontinuous at 0.

**At $x = 1$**

$$\lim_{x \to 1-} f(x) = \lim_{\substack{x \to 1 \\ (x<1)}} [x] = 0,$$

$$\lim_{x \to 1+} f(x) = \lim_{\substack{x \to 1 \\ (x>1)}} [x] = 1.$$

So $f$ discontinuous at 1.

Similarly, $f$ is discontinuous at 2, 3, 4, ...

**At $x = -1$.**

$$\lim_{x \to -1-} f(x) = \lim_{\substack{x \to -1 \\ (x<-1)}} [x] = -2,$$

$$\lim_{x \to -1+} f(x) = \lim_{\substack{x \to -1 \\ (x>-1)}} [x] = -1,$$

So $f$ is discontinuous at $-1$.

Let $\alpha \in \mathbf{R} \sim \mathbf{Z}$ be any real number but not an integer. Then there exists an integer $n$ such that $n \le \alpha < n+1$. Then

$$[x] = n, \text{ for } n \le x < n+1.$$

Now $\lim\limits_{x \to \alpha-} f(x) = \lim\limits_{x \to \alpha+} f(x) = f(\alpha) = n.$

Hence $f$ is continuous at $\alpha$.

**Example 27.** *Let f be the function on* $[0, 1]$ *defined by*

$$f(x) = \begin{cases} (-1)^r & \text{if } \frac{1}{r+1} \le x < \frac{1}{r}, \ r = 1, 2, \ldots \\ 0, & \text{if } x = 0 \\ 1 & \text{if } x = 1. \end{cases}$$

*Examine the continuity of f at* $1, \frac{1}{2}, \frac{1}{3}, \ldots$ [D.U., B.Sc. (G) 1992]

**Solution.** We observe that

$$\begin{aligned} f(x) &= -1, \text{ if } \tfrac{1}{2} \le x < 1 \quad (r=1) \\ &= 1, \text{ if } \tfrac{1}{3} \le x < \tfrac{1}{2} \quad (r=2) \\ &= -1, \text{ if } \tfrac{1}{4} \le x < \tfrac{1}{3} \quad (r=3) \end{aligned}$$

and so on.

**At $x = 1$.**

$$\lim_{x \to 1-} f(x) = \lim_{\substack{x \to 1 \\ (x<1)}} f(x) = -1. \text{ Also } f(1) = 1.$$

Since $\lim\limits_{x \to 1-} f(x) \ne f(1)$, $f$ is discontinuous at $x = 1$.

**At $x = \frac{1}{2}$.**

$$\lim_{x \to \frac{1}{2}-} f(x) = \lim_{\substack{x \to \frac{1}{2} \\ (x<\frac{1}{2})}} f(x) = 1,$$

$$\lim_{x \to \frac{1}{2}+} f(x) = \lim_{\substack{x \to \frac{1}{2} \\ (x>\frac{1}{2})}} f(x) = -1.$$

Thus $f$ is discontinuous at $\frac{1}{2}$.

Similarly, $f$ is discontinuous at $\frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$

We have

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0} (-x-1) = -1, \ \lim_{x \to 0+} f(x) = \lim_{x \to 0} x = 0\ ;$$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1} x = 1, \ \lim_{x \to 1+} f(x) = \lim_{x \to 1} (x+1) = 2\ ;$$

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2} (x+1) = 3, \ \lim_{x \to 2+} f(x) = \lim_{x \to 2} (x+2) = 4\ ;$$

$$\lim_{x \to -1-} f(x) = \lim_{x \to -1} (-x-2) = -3, \ \lim_{x \to -1+} f(x) = \lim_{x \to -1} (-x-2) = -2$$

and so on.

Thus $\lim_{x \to n-} f(x) \neq \lim_{x \to n+} f(x) \quad \forall \quad n \in \mathbf{Z}.$

Hence the points of discontinuity of $f$ are $0, \pm 1, \pm 2, \pm 3, \ldots$

**Example 30.** *Let f be a function defined by :*

$$f(x) = \begin{cases} x \sin(1/x) & , \text{ if } x > 0 \\ 0 & , \text{ if } x = 0 \\ 2x \sin(1/x) & , \text{ if } x < 0. \end{cases}$$

*Discuss the continuity of f at $x = 0$.* [D.U., B.Sc. (G) 1995]

**Solution.** We see that

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0} x \sin \frac{1}{x} = 0. \text{ [See Example 2]}$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0} 2x \sin \frac{1}{x} = 2 \times 0 = 0. \text{ Also } f(0) = 0.$$

Hence $f$ is continuous at $x = 0$.

**Example 31.** *Let f be defined on* **R** *by setting*

$$f(t) = \begin{cases} t & , \text{ if } 0 \le t \le < \frac{1}{2} \\ 0 & , \text{ if } t = \frac{1}{2} \\ t-1 & , \text{ if } \frac{1}{2} < t \le 1 \end{cases}$$

*and $f(n+t) = f(t)$, where n is any integer.*

*Determine the points of discontinuity of f.*

**Solution.** We shall show that $\{n + \frac{1}{2} : n \in \mathbf{Z}\}$ are the points of discontinuity of $f$.

Right hand limit of $f$ at $(n + \frac{1}{2}) = f(n + \frac{1}{2} + 0)$

$$= \lim f(n + \tfrac{1}{2} + h) = \lim_{\substack{h \to 0 \\ (h > 0)}} f\left(\tfrac{1}{2} + h\right) = \lim_{h \to 0} \left(\tfrac{1}{2} + h - 1\right) = -\tfrac{1}{2}.$$

$$\lim_{x \to -\frac{1}{2}+} f(x) = \lim_{x \to -\frac{1}{2}} \frac{2(-1)}{3x+x} = 1. \quad \text{Also } f(-\tfrac{1}{2}) = -1.$$

Hence $f$ is continuous at $x = -\frac{1}{2}$.

**Example 35.** *Let $f$ be the function defined on* **R** *by setting*

$$f(x) = x - [x] - \frac{1}{2}, \text{ when } x \text{ is not an integer,}$$

$$f(x) = 0 \quad , \text{ when } x \text{ is an integer.}$$

*Show that $f$ is continuous at all points of* **R** ~ **Z** *and is discontinuous whenever $x \in$* **Z**.

**Solution.** Let $n \in \mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ be arbitrary.

Then $\lim\limits_{x \to n} [x]$ does not exist and so $\lim\limits_{x \to n} f(x)$ does not exist.

Hence $f$ is discontinuous whenever $n \in \mathbf{Z}$.

We observe that

$$[x] = 0, \text{ if } 0 \le x < 1,$$
$$[x] = 1, \text{ if } 1 \le x < 2,$$
$$[x] = 2, \text{ if } 2 \le x < 3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[x] = m, \text{ if } m \le x \le m+1 \text{ and so on.}$$

Using these, we obtain

$$f(x) = x - \frac{1}{2}, \qquad \text{if } 0 \le x < 1,$$
$$f(x) = x - 1 - \frac{1}{2} = x - \frac{3}{2}, \qquad \text{if } 1 \le x < 2,$$
$$f(x) = x - 2 - \frac{1}{2} = x - \frac{5}{2}, \qquad \text{if } 2 \le x < 3 \text{ and so on.}$$

The above relations are all linear and so $f$ is continuous at all positive non-integral points. Similarly, $f$ is continuous at all negative non-integral points, since

$$f(x) = x - (-1) - \frac{1}{2} = x + \frac{1}{2}, \qquad \text{if } -1 \le x < 0$$
$$= x - (-2) - \frac{1}{2} = x + \frac{3}{2}, \qquad \text{if } -2 \le x < -1 \text{ etc.}$$

Hence $f$ is continuous at all points of **R** ~ **Z**.

## 5.6 Types of Discontinuities

Consider a function $f : [a, b] \to \mathbf{R}$ and $a < c < b$.

(*i*) *The function $f$ is said to have a* **removable discontinuity** *at $x = c$, if $\lim\limits_{x \to c} f(x)$ exists but is not equal to $f(c)$.*

(*ii*) *$f$ is said to have a* **discontinuity of the first kind** *at $x = c$, if $\lim\limits_{x \to c-} f(x)$ and $\lim\limits_{x \to c+} f(x)$ both exist but are not equal.*

Hence $f$ has a discontinuity of the first kind from the left and the right at $x = 0$.

**Example 39.** *A function $f$ is defined by*

$$\begin{aligned} f(x) &= -x^2 && \text{if} \quad x \leq 0, \\ &= 5x - 4 && \text{if} \quad 0 < x \leq 1, \\ &= 4x^2 - 3x && \text{if} \quad 1 < x < 2, \\ &= 3x + 4 && \text{if} \quad x \geq 2. \end{aligned}$$

*Examine $f$ for continuity at $x = 0, 1, 2$. Also discuss the kind of discontinuity, if any.*

**Solution.** *At* $x = 0$.

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0} (-x^2) = 0,$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0} (5x - 4) = -4.$$

Also $\qquad f(0) = 0.$

Thus $\lim\limits_{x \to 0+} f(x) \neq f(0)$ and so $f$ has a discontinuity of the first kind from the right at $x = 0$.

*At* $x = 1$.

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1} (5x - 4) = 1,$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1} (4x^2 - 3x) = 1,$$

and $\qquad f(1) = 5 \times 1 - 4 = 1.$

Thus $\lim\limits_{x \to 1-} f = \lim\limits_{x \to 1+} f(x) = f(1)$ and so $f$ is continuous at $x = 1$.

*At* $x = 2$.

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2} (4x^2 - 3x) = 4 \times 4 - 3 \times 2 = 10,$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2} (3x + 4) = 10.$$

Also $f(2) = 3 \times 2 + 4 = 10.$

Thus $\lim\limits_{x \to 2-} f(x) = \lim\limits_{x \to 2+} f(x) = f(x)$ and so $f$ is continuous at $x = 2$.

**Example 40.** *Obtain the points of discontinuity of the function $f$ defined on* [0, 1] *as follows :*

$$\begin{aligned} f(x) &= \tfrac{1}{2} - x, && \text{if} \quad 0 < x < \tfrac{1}{2} \\ &= \tfrac{3}{2} - x, && \text{if} \quad \tfrac{1}{2} < x < 1 \end{aligned}$$

$$f(0) = 0, f(\tfrac{1}{2}) = \tfrac{1}{2}, f(1) = 1.$$

*Also examine the kind of discontinuities.* **[D.U., B.A. (P) 1992]**

**Solution. At $x = 0$.**

$$\lim_{x\to 0+} f(x) = \lim_{\substack{x\to 0\\(x>0)}} f(x) = \lim_{x\to 0}\left(\tfrac{1}{2} - x\right) = \tfrac{1}{2}.$$

Also $\quad f(0) = 0. \quad \therefore \quad \lim_{x\to 0+} f(x) \neq f(0).$

Thus $f$ has a discontinuity of the first kind from the right at 0.

**At $x = \frac{1}{2}$.**

$$\lim_{x\to\frac{1}{2}-} f(x) = \lim_{x\to\frac{1}{2}}\left(\tfrac{1}{2} - x\right) = 0,$$

$$\lim_{x\to\frac{1}{2}+} f(x) = \lim_{x\to\frac{1}{2}}\left(\frac{3}{2} - x\right) = 1.$$

$$\therefore \quad \lim_{x\to\frac{1}{2}-} f(x) \neq \lim_{x\to\frac{1}{2}+} f(x).$$

Thus $f$ has a discontinuity of the first kind at $x = \frac{1}{2}$.

**At $x = 1$.**

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1}\left(\tfrac{3}{2} - x\right) = \tfrac{1}{2}. \quad \text{Also} \quad f(1) = 1.$$

$$\therefore \quad \lim_{x\to 1-} f(x) \neq f(1).$$

Thus $f$ has a discontinuity of the first kind from the left at $x = 1$.

**Example 41.** *Examine for continuity at $x = 0$, the function*

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1 + e^{1/x^2}}, & \textit{when } x \neq 0 \\ 0, & \textit{when } x = 0. \end{cases}$$

**Solution.** We know that

$$x \to 0- \Rightarrow e^{1/x} \to 0 \text{ and } e^{-1/x^2} \to 0,$$

and $$x \to 0+ \Rightarrow e^{-1/x} \to 0 \text{ and } e^{-1/x^2} \to 0.$$

[See Examples 11, 21]

$$\therefore \lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{e^{1/x}}{1 + e^{1/x^2}} = \lim_{x\to 0-} \frac{e^{1/x}\,.\,e^{-1/x^2}}{e^{-1/x^2} + 1} = \frac{0}{0+1} = 0 = f(0),$$

and $$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{e^{1/x}}{1 + e^{1/x^2}} = \lim_{x\to 0+} \frac{e^{-1/x^2}}{e^{-1/x}(e^{-1/x^2} + 1)} = \frac{0}{0}.$$

**It follows that $f$ is continuous from the left at $x = 0$. Further, since $\lim_{x\to 0+} f(x)$ does not exist, $f$ has a discontinuity of the second kind from the right at $x = 0$.**

**Example 42.** *Examine the function*

$$f(x) = \frac{e^{1/x}\sin(1/x)}{1 + e^{1/x}}, x \neq 0\,; f(0) = 0$$

*for points of discontinuity, if any.*

**Solution.** We know that

$$x \to 0- \Rightarrow e^{1/x} \to 0 \text{ and } x \to 0+ \Rightarrow e^{-1/x} \to 0.$$

$$\therefore \quad \lim_{x\to 0+} \frac{e^{1/x}\sin(1/x)}{1 + e^{1/x}} = \lim_{x\to 0+} \frac{\sin(1/x)}{e^{-1/x} + 1} = \lim_{x\to 0+} \sin\frac{1}{x},$$

which does not exist. Hence $f$ has a discontinuity of the second kind from the right at $x = 0$.

## EXERCISES

1. Show that the function

$$f(x) = 2 + x, \quad \text{if } x \geq 1$$
$$= 2 - x, \quad \text{if } x < 1$$

is discontinuous at $x = 1$.

2. Show that the function $f(x)$ defined as

$$f(x) = x, \quad \text{when } 0 \leq x < \frac{1}{2}$$
$$= 1, \quad \text{when } x = \frac{1}{2}$$
$$= 1 - x, \text{ when } \frac{1}{2} < x < 1$$

is discontinuous at $x = \frac{1}{2}$.

3. Let $f(x) = \dfrac{x^3 - 8}{x^2 - 4}, x \neq 2$

$$= 3, \quad x = 2.$$

Show that $f(x)$ is continuous at $x = 2$.

4. Examine the following functions for continuity at $x = 0$:

(*i*) $f(x) = \begin{cases} e^{1/x}, & \text{when } x \neq 0 \\ 0\,, & \text{when } x = 0. \end{cases}$

(*ii*) $f(x) = \begin{cases} e^{-1/x}, & \text{when } x \neq 0 \\ 0\,, & \text{when } x = 0. \end{cases}$

(*iii*) $f(x) = \begin{cases} \dfrac{1}{1 - e^{1/x}}, & \text{when } x \neq 0 \\ 0\,, & \text{when } x = 0. \end{cases}$

[**Hint.** (*i*) $\lim_{x\to 0-} f(x) = 0 = f(0)$ and $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} 1/e^{-1/x} = \infty.$

Hence $f$ is continuous from the left at $x = 0$ and $f$ has discontinuity of the second kind from the right at $x = 0$.

(*ii*) $f$ has discontinuity of the second kind from the left at $x = 0$.

(*iii*) Since $\lim\limits_{x \to 0-} f(x) = 1 \neq f(0)$, $f$ has discontinuity of the first kind from the left at $x = 0$. $f$ is continuous elsewhere.]

**5.** Show that the function $f$ defined as

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1 + e^{1/x}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is not continuous at $x = 0$.

**6.** Show that $f(x) = \dfrac{x - 1}{1 + e^{1/(x-1)}}$, $f(1) = 0$

is continuous at $x = 1$.

[**Hint.** Let $x - 1 = y$ so that $y \to 0$ as $x \to 1$.

$$\lim_{x \to 1-} f(x) = \lim_{y \to 0-} \frac{y}{1 + e^{1/y}} = \frac{0}{1 + 0} = 0,$$

$$\lim_{y \to 1+} f(x) = \lim_{y \to 0+} \frac{y}{1 + e^{1/y}} = \lim_{y \to 0+} \frac{y\, e^{-1/y}}{e^{-1/y} + 1} = \frac{0.0}{0 + 1} = 0.]$$

**7.** Discuss continuity of the function

$$f(x) = \begin{cases} x\, e^{1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

at $x = 0$. [D.U., B.C.A. 1998]

[**Hint.** $\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0-} x\, e^{1/x} = 0 \times 0 = 0,$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x / e^{-1/x} = \frac{0}{0}.$$

Hence $f$ has discontinuity of the second kind from the right at $x = 0$.]

**8.** Give examples of functions $f$ and $g$ such that $f$, $g$ and $fg$ are discontinuous while $f + g$ is a continuous function. [D.U., B.C.A. 1999]

[**Hint.** Let $f(x) = [x]$ and $g(x) = x - [x]$ $\forall$ $x \geq 0$. Then $f, g$ are discontinuous at 0, 1, 2, 3, ... and $(fg)(x) = [x]\,(x - [x])$ $\forall x \geq 0$ is discontinuous at $x = 2, 3, \ldots$; since

$(fg)(x) = 0$, if $0 \leq x < 1$; $(fg)(x) = 1\,(x - 1)$, if $1 \leq x < 2$;

$(fg)(x) = 2\,(x - 2)$, if $2 \leq x < 3$; $(fg)(x) = 3\,(x - 3)$, if $3 \leq x < 4$ etc.

Thus $f, g, fg$ are not continuous $\forall$ $x \geq 0$, but

$$(f + g)(x) = x \text{ is continuous } \forall \ x \geq 0.$$

**9.** Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{x + |x|}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

at $x = 0$. [D.U., B.Sc. (G) 1997]

[**Hint.** $\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0} \dfrac{x - x}{-x} = 0$, $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0} \dfrac{x + x}{x} = 2$.

Hence $f$ is discontinuous at $x = 0$.]

**10.** Examine the continuity of the function

$$f(x) = \begin{cases} \dfrac{3x + |x|}{7x - 5|x|}, & x = 0 \\ 0, & x = 0 \end{cases}$$

at $x = 0$. (**Ans.** Discontinuous)

**11.** Let $f$ be the function defined on [0, 1] by setting

$$f(x) = \frac{1}{2^n}, \text{ when } \frac{1}{2^{n+1}} \leq x < \frac{1}{2^n} \ (n = 0, 1, 2, \ldots)$$

Show that $f$ is discontinuous at the points $(\frac{1}{2})^n$ $(n = 1, 2, 3, \ldots)$. Describe the nature of discontinuity at each of these points.

[**Hint.** Refer to Example 28.

$$\lim_{x \to (\frac{1}{2})^n -} f(x) = \frac{1}{2^n} \text{ and } f\left(\frac{1}{2^n}\right) = \frac{1}{2^{n-1}} \text{ for } n = 1, 2, 3, \ldots$$

Thus $f$ has discontinuity of the first kind from the left at the points $(\frac{1}{2})^n$, $n = 1, 2, 3, \ldots$.]

**12.** Let $f$ be the function defined on [0, 1] by setting

$$f(x) = 2rx, \quad \text{when } \frac{1}{r + 1} \leq x < \frac{1}{r} \ (r = 1, 2, 3, \ldots)$$

$f(0) = 0$, $f(1) = 1$. Examine for continuity the function $f$ at the points $1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{r}, \ldots, 0$.

[**Hint.** $\lim\limits_{x \to (1/r)-} f(x) = \lim\limits_{x \to 1/r} 2rx = 2$, $r = 1, 2, 3, \ldots$

$$f(1/r) = 2\,(r - 1)/r \text{ for } r = 2, 3, \ldots \text{ and } f(1) = 1.$$

Thus $f$ has discontinuity of the first kind from the left at $1/r$, $r = 1, 2, 3, \ldots$. Since $\lim\limits_{x \to 0+} f(x)$ does not exist, $f$ has discontinuity of the second kind from the right at 0.]

**13.** A function $f$ is defined on [0, 1] by

$$f(x) = \frac{1}{n}, \text{ for } \frac{1}{n + 1} < x \leq \frac{1}{n}, \ n = 1, 2, 3, \ldots$$

Show that the points of discontinuity of $f$ are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$

**14.** Let $f$ be the function defined on **R** by setting

$$f(x) = [x] + [-x], \text{ for all } x \in \mathbf{R}.$$

Find the points of discontinuity of $f$. (**Ans.** $0, \pm 1, \pm 2, \ldots$)

[**Hint.** Similar to **Example 29.** We have

$$f(x) = 0 - 1 = -1, \text{ for } 0 < x < 1 ; f(1) = 0$$
$$f(x) = 1 - 2 = -1, \text{ for } 1 < x < 2 ; f(2) = 0$$
$$f(x) = 2 - 3 = -1, \text{ for } 2 < x < 3 ; f(3) = 0 \text{ etc.}$$
$$f(x) = -1 + 1 = 0, \text{ for } -1 < x < 0 ; f(-1) = 0$$
$$f(x) = -2 + 2 = 0, \text{ for } -2 < x < -1 ; f(-2) = 0 \text{ etc.]}$$

**15.** Examine the function : $f(x) = (x - [x])^2, x \geq 0$ for points of discontinuity and the nature of discontinuity.

[**Hint.** $f(x) = (x - 0)^2$, for $0 \leq x < 1$ ; $f(x) = (x - 1)^2$, for $1 \leq x < 2$ etc.

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1} (x - 0)^2 = 1, f(1) = (1 - 1)^2 = 0.$$

Thus $f$ has discontinuity of the first kind from left at $x = 1$.

Similar result holds for $x = 2, 3, 4, \ldots$ Further $f$ is continuous at all points other than $1, 2, 3, 4, \ldots$

**16.** Discuss the continuity of the function $f$ defined as :

$f(x) = [x] + (x - [x])^2$, for all $x \geq 0$. (**Ans.** Continuous for all $x \geq 0$)

[**Hint.** We have

$$f(x) = 0 + (x - 0)^2 = x^2, 0 \leq x < 1 ; f(x) = 1 + (x - 1)^2, 1 \leq x < 2$$
$$\lim_{x \to 1-} f(x) = \lim_{x \to 1} x^2 = 1, \lim_{x \to 1+} f(x) = \lim_{x \to 1} 1 + (x - 1)^2 = 1.$$

Also $f(1) = 1$. Hence $f$ is continuous at $x = 1$. Similarly, $f$ is continuous at $x = 0, 2, 3, 4, \ldots$ Indeed, $f$ is continuous for all $x \geq 0$.]

**17.** Show that the function $f(x)$ defined on $[0, 1]$, as follows, is discontinuous at $x = 0, \frac{1}{2}$ and $1$ :

$$f(0) = 2 ; f(x) = x + \tfrac{1}{2}, 0 < x < \tfrac{1}{2} ; f(\tfrac{1}{2}) = \tfrac{1}{2} ;$$
$$f(x) = 3x - \tfrac{1}{2}, \tfrac{1}{2} < x < 1 \text{ and } f(1) = 1.$$

Also mention the type of discontinuity at each point.

[D.U., B.C.A. 1998]

[**Hint.** $f$ has discontinuity of the first kind from the right at $x = 0$ ; $f$ has a removable discontinuity at $x = \frac{1}{2}$ ; $f$ has discontinuity of the first kind from the left at $x = 1$.]

**18.** Let $f$ be defined on **R** as follows :

$$f(x) = \lim_{n \to \infty} \frac{(1 + \sin \pi x)^n - 1}{(1 + \sin \pi x)^n + 1}.$$

Show that $f$ is discontinuous at integral values of $x$.

[**Hint.** Clearly, $f(x) = 0$, if $x$ is an integer. ($\because \sin \pi x = 0$)

Let $2m < x < 2m + 1$ ($m$ an integer) $\Rightarrow 2m\pi < \pi x < (2m + 1)\pi \Rightarrow$
$0 < \sin \pi x < 1 \Rightarrow 1 < 1 + \sin \pi x < 2$
$\Rightarrow (1 + \sin \pi x)^n \to \infty$ as $n \to \infty \Rightarrow (1 + \sin \pi x)^{-n} \to 0$ as $n \to \infty$
$\Rightarrow f(x) = 1$.
Let $2m - 1 < x < 2m \Rightarrow (2m - 1)\pi < \pi x < 2m\pi \Rightarrow -1 < \sin \pi x < 0$
$\Rightarrow 0 < 1 + \sin \pi x < 1 \Rightarrow (1 + \sin \pi x)^n \to 0$ as $n \to \infty \Rightarrow f(x) = -1$.

$$\therefore \quad f(x) = \begin{cases} 0, & \text{if } x = 0, \pm 1, \pm 2, \ldots \\ -1, & \text{if } 2m - 1 < x < 2m \\ 1, & \text{if } 2m < x < 2m. \end{cases} \quad (m \text{ an integer})$$

Now show that $f$ is discontinuous at $2m, 2m - 1, m \in \mathbf{Z}$ ]

## 5.7 Theorems on Continuous Functions

**Theorem 7.** (*a*) *If two functions $f, g$ are continuous at a point $c$, then the functions $f + g, f - g, fg$ are also continuous at $c$ and if $g(c) \neq 0$, then $f/g$ is also continuous at $c$.*

(*b*) *Prove that if a function $f$ is continuous at $x = a$, then $|f|$ is also continuous at $x = a$. Show that the converse is not true.*

[D.U., Maths (H) 1998]

**Proof.** Since $f$ and $g$ are continuous at $c$,

$$\lim_{x \to c} f(x) = f(c) \text{ and } \lim_{x \to c} g(x) = g(c). \qquad \ldots(1)$$

We show that $f + g$ is continuous at $c$. We have

$$\lim_{x \to c} (f + g)(x) = \lim_{x \to c} [f(x) + g(x)]$$
$$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \qquad \text{[Theorem 2]}$$
$$= f(c) + g(c), \text{ by } (1)$$

$$\therefore \quad \lim_{x \to c} (f + g)(x) = (f + g)(c).$$

Thus $f + g$ is continuous at $c$.

Again, $\lim_{x \to c} (fg)(x) = \lim_{x \to c} f(x) g(x)$

$$= \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) \qquad \text{[Theorem 2]}$$
$$= f(c) g(c) = (fg)(c).$$

Thus $fg$ is continuous at $c$

Similarly, we can prove the remaining parts.

(*b*) Since $f$ is continuous at $x = a$, therefore, for $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{when} \quad 0 < |x - a| < \delta. \qquad \ldots(2)$$

We know $||x|-|y|| \leq |x-y| \quad \forall \quad x, y \in \mathbf{R}.$

$\therefore \quad ||f(x)|-|f(a)|| \leq |f(x)-f(a)| < \varepsilon, 0 < |x-a| < \delta,$ using (2)

*i.e.,* $\quad ||f(x)|-|f(a)|| < \varepsilon,$ when $0 < |x-a| < \delta.$

Hence $|f|$ is continuous at $x = a$.

*However, if $|f|$ is continuous at $x = a$, then $f$ may not be continuous at $x = a$* as seen below :

$$\text{Let} \quad f(x) = \begin{cases} 1, & \text{when } x \text{ is rational}, \\ -1, & \text{when } x \text{ is irrational}. \end{cases}$$

Then $|f(x)| = 1 \ \forall \ x \in \mathbf{R}$ and so $|f|$ is continuous at every point of $\mathbf{R}$, but $f$ is discontinuous at every point of $\mathbf{R}$. [See Example 43 ahead]

**Remarks. 1.** *Let $f$ and $g$ be defined on an interval $I$. If $f+g$ and $fg$ are continuous at a point $p \in I$, then $f$ and $g$ may not be continuous at $p$ as explained by the following examples :*

(*i*) Let $\quad f(x) = \dfrac{e^{1/x}-1}{e^{1/x}+1}, x \neq 0$ and $f(0) = 0$; [See Example 18]

$$g(x) = \frac{1-e^{1/x}}{e^{1/x}+1}, x \neq 0 \text{ and } g(0) = 0.$$

Thus $f$ and $g$ are discontinuous at $x = 0$, but $f+g = 0$ (being a constant function) is continuous at $x = 0$.

(*ii*) Let $f(x) = g(x) = 1, \quad$ when $x$ is rational,

$\qquad f(x) = g(x) = -1, \quad$ when $x$ is irrational.

By Example 43, $f$ and $g$ are discontinuous at every point of $\mathbf{R}$, but $fg = 1$ (being a constant function) is continuous at every point of $\mathbf{R}$.

*Similarly, we can give examples to show that if $f-g$, $f/g$ are continuous at a point $p \in I$ ; then $f$ and $g$ may not be continuous at $p$.*

Notice that in (*ii*), $f/g = 1 \ \forall \ x \in \mathbf{R} \Rightarrow f/g$ is continuous $\forall \ x \in \mathbf{R}$.

If in (*i*), $g(x) = f(x)$, then $f-g = 0 \Rightarrow f-g$ is continuous $\forall \ x \in \mathbf{R}$.

**2.** *A polynomial function $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is continuous for all $x \in \mathbf{R}$.* [D.U., B.C.A. 1998]

Clearly, $\lim\limits_{x \to c} x^n = c^n \ \forall \ n \in \mathbf{N} \Rightarrow x^n$ is continuous $\forall \ x \in \mathbf{R}, \forall \ n \in \mathbf{N}$.

Also $a_n$ (being a constant) is continuous. By Theorem 7, $a_nx^n$ is continuous $\forall \ n \in \mathbf{N}$ and $\forall \ x \in \mathbf{R}$. Hence $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is continuous $\forall \ x \in \mathbf{R}$ (Theorem 7).

**Theorem 8.** *A function $f$ defined on an interval $I$ is continuous at a point $p \in I$ if and only if for each sequence $\langle p_n \rangle$ converging to $p$, the sequence $\langle f(p_n) \rangle$ converges to $f(p)$.* [D.U., Maths (H) 1999, 95, 92]

**Proof.** *The condition is necessary.*

We are given that the function $f$ is continuous at $p$ and the sequence $\langle p_n \rangle$ converges to $p$. We shall show that $\langle f(p_n) \rangle$ converges to $f(p)$. Since $f$ is continuous at $x = p$, therefore for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(p)| < \varepsilon, \quad \text{when} \quad |x-p| < \delta. \qquad \dots(1)$$

Since the sequence $\langle p_n \rangle$ converges to $p$, so there exists some positive integer $m$ such that

$$|p_n - p| < \delta \quad \forall \quad n \geq m. \qquad \dots(2)$$

Replacing $x$ by $p_n$ in (1), we get

$$|f(p_n) - f(p)| < \varepsilon \quad \text{when} \quad |p_n - p| < \delta. \qquad \dots(3)$$

From (2) and (3), we obtain

$$|f(p_n) - f(p)| < \varepsilon \quad \forall \ n \geq m. \quad \text{Thus } \langle f(p_n) \rangle \to f(p).$$

*The condition is sufficient.*

Let $\ . \ \langle p_n \rangle \to p \ \Rightarrow \ \langle f(p_n) \rangle \to f(p)$.

We shall show that $f$ is continuous at $x = p$.

Let, if possible, $f$ be not continuous at $x = p$.

Then for some $\varepsilon > 0$ and for each $\delta > 0$, $\exists$ at least one $x$ such that

$$|f(x) - f(p)| \geq \varepsilon, \quad \text{whenever} \quad |x-p| < \delta.$$

[See Remark 1, Sec. 5.5]

Let $\delta = 1/n$. Then $\forall \ n \in \mathbf{N}$, there exists $x = p_n$ such that

$$|p_n - p| < 1/n \quad \Rightarrow \quad |f(p_n) - f(p)| \geq \varepsilon$$

*i.e.,* $\quad \lim\limits_{n \to \infty} p_n = p$, but $\lim\limits_{n \to \infty} f(p_n) \neq f(p)$. $\quad (\because \ p_n - p \to 0 \text{ as } n \to \infty)$

Thus we have proved that there is a sequence $\langle p_n \rangle$ such that $\langle p_n \rangle$ converges to $p$, but the sequence $\langle f(p_n) \rangle$ does not converge to $f(p)$. This is a contradiction to the given condition. Hence $f$ is continuous at $x = p$.

**Theorem 9.** *Let $f$ be a function defined on an interval $I$ and $p \in I$. Let $g$ be a function defined on an interval $J$ such that $f(I) \subseteq J$. If $f$ is continuous at $p$ and $g$ is continuous at $f(p)$, then show that $gof$ is continuous at $p$.*

**Proof.** Let $\langle p_n \rangle$ be any sequence in $I$ such that $p_n \to p$. $\qquad \dots(1)$

Since $f$ is continuous at $p$, so by Theorem 8, $f(p_n) \to f(p)$.

Since $f(I) \subseteq J$, so $\langle f(p_n) \rangle$ is a sequence in $J$ converging to $f(p) \in J$.

Since $g$ is continuous at $f(p)$ and $f(p_n) \to f(p)$ in $J$, so

$$g(f(p_n)) \to g(f(p)) \qquad \dots(2)$$

or $\qquad (gof)(p_n) \to (gof)(p)$.

[By definition, $(gof)(x) = g(f(x)) \ \forall \ x \in I$.]

From (1) and (2), it follows that $gof$ is continuous at $p$ (By Theorem 8)

## EXAMPLES

**Example 43.** *Let $f$ be a function on* **R** *defined by*

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational.} \end{cases}$$

*Show that $f$ is discontinuous at every point of* **R**.

[D.U., Maths (H) 1999, 97, 93 ; B.A. (P) 1998 ; B.Sc. (G) 1996, 93]

**Solution. Case I.** Let $a$ be any rational number, so that $f(a) = 1$.

Then any *nbd.* $\left]a - \frac{1}{n}, a + \frac{1}{n}\right[$ of $a$ contains an *irrational number* $a_n$ for each $n \in \mathbf{N}$ *i.e.*, $a_n \in \left]a - \frac{1}{n}, a + \frac{1}{n}\right[ \quad \forall \quad n \in \mathbf{N}$

$$\Rightarrow \quad a - \frac{1}{n} < a_n < a + \frac{1}{n} \quad \forall \quad n \in \mathbf{N}$$

$$\Rightarrow \quad |a_n - a| < \frac{1}{n} \quad \forall \quad n \in \mathbf{N} \Rightarrow |a_n - a| \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \quad a_n - a \to 0 \quad \Rightarrow \quad \text{as } n \to \infty \Rightarrow \langle a_n \rangle \text{ converges to } a.$$

Now $\quad f(a_n) = -1 \quad$ for all $n \qquad (\because\ a_n$ is irrational) and $f(a) = 1$.

$$\therefore \quad \lim_{n \to \infty} f(a_n) = -1 \neq f(a) \text{ i.e., } \langle f(a_n) \rangle \text{ does not converge to } f(a),$$

where $a_n \to a$. Thus $f$ is discontinuous at all rational points $a$.

**Case II.** Let $b$ be any irrational number so that $f(b) = -1$.

As argued above, we can choose a *rational number* $b_n$ such that

$$|b_n - b| < \frac{1}{n} \quad \Rightarrow \quad b_n \to b \text{ as } n \to \infty$$

$$\Rightarrow \quad \langle b_n \rangle \text{ converges to } b.$$

Now $\quad f(b_n) = 1 \quad$ for all $n \qquad (\because\ b_n$ is rational) and $f(b) = -1$.

$$\therefore \quad \lim_{n \to \infty} f(b_n) = 1 \neq f(b).$$

$\Rightarrow \langle f(b_n) \rangle$ does not converge to $f(b)$, where $b_n \to b$.

Thus the function $f$ is discontinuous at all irrational points $b$.

Hence the given function is discontinuous at all points of **R**.

**Example 44.** *Show that the function $f$ defined by*

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

*is discontinuous at every point.* [D.U., Maths (H) 1989]

**Solution. Case I.** Let $a$ be any rational number so that $f(a) = 1$.

Then any *nbd.* $]a - 1/n, a + 1/n[$ of $a$ contains an *irrational number* $a_n$ for each $n \in \mathbf{N}$ *i.e.*,

$$a_n \in \,]a - 1/n, a + 1/n[\, \Rightarrow |a_n - a| < \frac{1}{n}$$

$$\Rightarrow |a_n - a| \to 0 \text{ as } n \to \infty \Rightarrow \langle a_n \rangle \to a.$$

Now $f(a_n) = 0 \ \ \forall \ \ n \quad (\because\ a_n$ is irrational) and $f(a) = 1$

$$\Rightarrow \quad \lim_{n \to \infty} f(a_n) = 0 \neq f(a) \quad \Rightarrow \quad \langle f(a_n) \rangle \text{ does not converge to } f(a).$$

Thus $f$ is discontinuous at all rational points.

**Case II.** Let $b$ be any irrational number so that $f(b) = 0$.

As argued earlier, we can choose a *rational number* $b_n$ such that

$$|b_n - b| < \frac{1}{n} \ \forall \ n \in \mathbf{N}$$

$$\Rightarrow |b_n - b| \to 0 \text{ as } n \to \infty \quad \Rightarrow \quad \langle b_n \rangle \to b.$$

Now $f(b_n) = 1 \ \ \forall \ \ n \ \ (\because\ b_n$ is rational) and $f(b) = 0$

$$\Rightarrow \quad \lim_{n \to \infty} f(b_n) = 1 \neq f(b) \Rightarrow \langle f(b_n) \rangle \text{ does not converge to } f(b).$$

Thus $f$ is discontinuous at all irrational points.

Hence the given function is discontinuous at every point of **R**.

**Example 45.** *Show that the function $f$ defined on* **R** *by*

$$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational} \end{cases}$$

*is continuous only at $x = 0$.* [D.U., Maths (H) 1996, 97]

**Solution. Case I.** Let $a \neq 0$ be any rational number so that

$$f(a) = -a.$$

Then any *nbd.* $]a - 1/n, a + 1/n[$ of $a$ contains an *irrational number* $a_n$ for each $n \in \mathbf{N}$ *i.e.*,

$$a_n \in \,]a - 1/n, a + 1/n[\, \quad \Rightarrow \quad |a_n - a| < \frac{1}{n}$$

$$\Rightarrow \quad |a_n - a| \to 0 \text{ as } n \to \infty \quad \Rightarrow \quad \langle a_n \rangle \to a.$$

Now $\qquad f(a_n) = a_n \ \forall \ n \qquad (\because\ a_n$ is irrational)

$$\Rightarrow \quad \lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} a_n = a \neq f(a) \qquad (\because\ f(a) = -a)$$

$\Rightarrow \langle f(a_n) \rangle$ does not converge to $f(a)$, where $\langle a_n \rangle \to a$.

So $f$ is discontinuous at all non-zero rational points.

**Case II.** Let $b \neq 0$ be any irrational number so that $f(b) = b$.

As argued earlier, there exists a *rational number* $b_n \ \forall \ n \in \mathbf{N}$ such that

$$|b_n - b| < \frac{1}{n} \quad \Rightarrow |b_n - b| \to 0 \text{ as } n \to \infty \quad \Rightarrow \langle b_n \rangle \to b.$$

$$\text{Now} \quad f(b_n) = -b_n \quad \forall\ n \in \mathbf{N} \qquad (\because\ b_n \text{ is a rational number})$$

$$\Rightarrow \quad \lim_{n \to \infty} f(b_n) = -\lim_{n \to \infty} b_n = -b \neq f(b) \qquad (\because\ f(b) = b)$$

Thus $\langle b_n \rangle \to b$, but $\langle f(b_n) \rangle$ does not converge to $f(b)$.

So $f$ is discontinuous at all non-zero irrational points.

We shall now prove that $f$ is continuous only at the point $x = 0$.

Let $\varepsilon > 0$ be any number, and $\delta = \varepsilon/2$. Then

$$|x| < \delta \Rightarrow |f(x) - f(0)| = |x| < \delta < \varepsilon, \text{ when } x \text{ is irrational}$$

$$\text{and } |x| < \delta \Rightarrow |f(x) - f(0)| = |-x| = |x| < \delta < \varepsilon, \text{ when } x \text{ is rational.}$$

$$\therefore \quad |f(x) - f(0)| < \varepsilon, \text{ whenever } |x - 0| < \delta.$$

Hence the function $f$ is continuous only at $x = 0$.

**Example 46.** *Show that the function f defined by*

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

*is continuous only at* $x = 0$. [D.U., B.A. (P) 1997]

**Solution. Case I.** Let $a \neq 0$ be any rational number, so that

$$f(a) = a.$$

Then any *nbd.* $]a - 1/n, a + 1/n[$ of $a$ contains an *irrational number* $a_n$ for each $n \in \mathbf{N}$ *i.e.*,

$$a_n \in\ ]a - 1/n, a + 1/n[ \quad \Rightarrow \quad |a_n - a| < \frac{1}{n}$$

$$\Rightarrow \quad |a_n - a| \to 0 \text{ as } n \to \infty \quad \Rightarrow \quad \langle a_n \rangle \to a.$$

$$\text{Now} \quad f(a_n) = 0\ \forall\ n \qquad (\because\ a_n \text{ is irrational})$$

$$\Rightarrow \quad \lim_{n \to \infty} f(a_n) = 0 \neq f(a) \quad (\because\ f(a) = a \neq 0)$$

Thus $f$ is discontinuous at every non-zero rational point.

**Case II.** Let $b \neq 0$ be any irrational number, so that $f(b) = 0$.

Then there exists a *rational number* $b_n$ $\forall\ n \in \mathbf{N}$ such that

$$|b_n - b| < \frac{1}{n} \Rightarrow |b_n - b| \to 0 \text{ as } n \to \infty \Rightarrow \langle b_n \rangle \to b.$$

$$\text{Now} \quad f(b_n) = b_n \quad \forall\ n \qquad (\because\ b_n \text{ is rational})$$

$$\Rightarrow \quad \lim_{n \to \infty} f(b_n) = \lim_{n \to \infty} b_n = b \neq 0$$

$$\Rightarrow \quad \langle f(b_n) \rangle \text{ does not converge to } f(b). \qquad (\because\ f(b) = 0)$$

Thus $f$ is discontinuous at every non-zero irrational point.

Now we shall prove that $f$ is continuous at $x = 0$.

We have $f(0) = 0$ and

$$|f(x) - f(0)| = |f(x)| = |x|, \text{ if } x \text{ is rational}$$
$$= 0, \text{ if } x \text{ is irrational}$$

Let $\varepsilon > 0$. Then $|f(x) - f(0)| < \varepsilon$, for $|x - 0| < \delta\ (\delta = \varepsilon)$.

Hence $f$ is continuous only at $x = 0$.

**Example 47.** (*a*) *Let f be the function defined on* $[-1, 1]$ *by*

$$f(x) = \begin{cases} x, & \text{if } x \text{ is irrational} \\ 0, & \text{if } x \text{ is rational.} \end{cases}$$

*Prove that f is continuous only at* $x = 0$. [D.U., B.Sc. (G) 1990]

(*b*) *Show that a function f defined on* $[0, 1]$ *by*

$$f(x) = \begin{cases} x, & \text{when } x \text{ is rational} \\ 2x, & \text{when } x \text{ is irrational} \end{cases}$$

*is continuous only at* $x = 0$. [D.U., Physics (H) 1998]

**Please try yourself.**

**Example 48.** *Let f be a function defined on* $]0, 1[$ *by* :

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1/q, & \text{if } x = p/q, \text{ where } p \text{ and } q \text{ are positive integers having no common factor.} \end{cases}$$

*Prove that f is continuous at each irrational point and discontinuous at each rational point.*

**Solution. Case I.** Let $a \in\ ]0, 1[$ be any rational number so that $a = p/q$, where $p$ and $q$ are positive integers having no common factor. Then $f(a) = 1/q$ (as given).

Any *nbd.* $]a - 1/n, a + 1/n[$ of $a$ contains an *irrational number* $a_n$ for each $n \in \mathbf{N}$ *i.e.*,

$$a_n \in\ ]a - 1/n, a + 1/n[ \quad \Rightarrow \quad |a_n - a| < \frac{1}{n}$$

$$\Rightarrow \quad |a_n - a| \to 0 \text{ as } n \to \infty \quad \Rightarrow \quad \langle a_n \rangle \to a.$$

$$\text{Now} \quad f(a_n) = 0 \quad \forall\ n \qquad (\because\ a_n \text{ is irrational})$$

$$\Rightarrow \quad \langle f(a_n) \rangle \to 0 \neq f(a) \quad [\because\ f(a) = 1/q > 0.]$$

Hence $f$ is discontinuous at every rational point in $]0, 1[$.

**Case II.** Let $b \in\ ]0, 1[$ be any irrational number, so that

$$f(b) = 0.$$

Let $\varepsilon > 0$ be given. We can choose a positive integer $n$ such that

$$1/n < \varepsilon.$$

It is clear that there can be only a finite number of rational numbers $p/q$ in $]0, 1[$ such that $q < n$.

We have $|f(x)-f(\frac{1}{2})|=|x-\frac{1}{2}|$, if $x$ is rational $\quad (\because f(\frac{1}{2})=\frac{1}{2})$

$$=|1-x-\tfrac{1}{2}|=|\tfrac{1}{2}-x|=|x-\tfrac{1}{2}|,$$

if $x$ is irrational.

Thus $\quad |f(x)-f(\frac{1}{2})|=|x-\frac{1}{2}|$.

Let $\varepsilon>0$ be given. Choose $\delta=\varepsilon>0$. Then

$$|f(x)-f(\tfrac{1}{2})|<\varepsilon, \quad \text{when} \quad |x-\tfrac{1}{2}|<\delta.$$

Hence $f$ is continuous only at $\frac{1}{2}$.

**Example 50.** *Let $f$ be continuous on $[a, b]$ and let $f(x)=0$ for all $x\in \mathbf{Q}$. Show that $f(x)=0\ \forall\ x\in[a, b]$.*

**Solution.** Let $r$ be any rational number in $[a, b]$. Then $f(r)=0$.

Let $x$ be *any* irrational number in $[a, b]$.

Then any *nbd.* $]x-1/n, x+1/n[$ of $x$ contains a *rational number* $x_n$ for each $n\in\mathbf{N}$ *i.e.*,

$$x_n\in\left]x-\frac{1}{n}, x+\frac{1}{n}\right[ \ \Rightarrow\ |x_n-x|<\frac{1}{n}$$

$$\Rightarrow \quad |x_n-x|\to 0 \text{ as } n\to\infty \quad \Rightarrow\ \langle x_n\rangle\to x.$$

Since $f$ is continuous on $[a, b]$ and $\langle x_n\rangle\to x$, where $x\in[a, b]$ ;

$$\therefore \quad \langle f(x_n)\rangle\to f(x) \text{ or } \lim_{n\to\infty} f(x_n)=f(x). \qquad \text{...(1)}$$

$$\text{Since } x_n\in\mathbf{Q},\ f(x_n)=0\ \forall\ n \text{ and so } \lim_{n\to\infty} f(x_n)=0. \qquad \text{...(2)}$$

From (1) and (2), $f(x)=0$ for all irrational $x\in[a, b]$.

Hence $\quad f(x)=0\quad\forall\quad x\in[a, b]$.

**Example 51.** *Let $f$ be continuous in* **R**. *Show that the set*

$$A=\{x: f(x)=0\} \text{ is closed.}$$

**Solution.** Let $p$ be any limit point of $A$. We have to show that $p\in A$. By definition of a limit point, any *nbd.* $]p-1/n, p+1/n[$ of $p$ contains $p_n\in A$ for infinitely many values of $n$, say $n\geq m$ for some $m\in\mathbf{N}$.

$\Rightarrow p_n\in\ ]p-1/n, p+1/n[$, where $p_n\in A$ and $n\geq m$

$\Rightarrow |p_n-p|<1/n$, where $f(p_n)=0$ and $n\geq m$

$\Rightarrow p_n-p\to 0$ as $n\to\infty$, where $\lim\limits_{n\to\infty} f(p_n)=0$. ...(1)

Now $\quad p_n\to p\ \Rightarrow\ f(p_n)\to f(p)$, since $f$ is continuous at $p\in\mathbf{R}$

$\Rightarrow \lim\limits_{n\to\infty} f(p_n)=f(p)\ \Rightarrow\ f(p)=0$, using (1).

Now $\quad f(p)=0 \quad\Rightarrow\ p\in A$. Hence $A$ is a closed set.

$$\Rightarrow \qquad x_n - x \to 0$$

$$\Rightarrow \qquad f(x_n - x) \to 0, \text{ using (4)}$$

$$\Rightarrow \qquad f(x_n) + f(-x) \to 0, \text{ using (1)}$$

$$\Rightarrow \qquad f(x_n) - f(x) \to 0, \text{ using (3)}$$

$$\Rightarrow \qquad f(x_n) \to f(x).$$

Since $x_n \to x \Rightarrow f(x_n) \to f(x)$, $f$ is continuous for all $x \in \mathbf{R}$.

**Example 55.** *Prove that the function* $h(x) = \sqrt{\sin x}$ *is continuous on* $[0, \pi]$.

**Solution.** First of all, we show that $f(x) = \sqrt{x}$ is continuous $\forall\ x \ge 0$.

Let $c > 0$ so that $f(c) = \sqrt{c}$. For $x \ge 0$, we have

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| = \left| \frac{x - c}{\sqrt{x} + \sqrt{c}} \right| < \frac{1}{\sqrt{c}} |x - c|.$$

Let $\varepsilon > 0$ be given. Then

$$|f(x) - f(c)| < \varepsilon, \quad \text{when} \quad |x - c| < \delta, \delta = \sqrt{c}\, \varepsilon > 0.$$

Hence $f$ is continuous at $c > 0$.

Obviously, $\lim\limits_{x \to 0} f(x) = 0 = f(0)$. Hence $f$ is continuous for all $x \ge 0$.

Next we show that $g(x) = \sin x$ is continuous for all $x \in \mathbf{R}$.

For $c \in \mathbf{R}$, we have

$$|g(x) - g(c)| = |\sin x - \sin c| = 2 \left| \cos \frac{x+c}{2} \sin \frac{x-c}{2} \right|$$

$$\le 2 \left| \sin \frac{x-c}{2} \right|, \text{ since } |\cos x| \le 1\ \forall\ x \in \mathbf{R}$$

$$\le 2 \left| \frac{x-c}{2} \right|, \quad \text{since } |\sin x| \le |x|$$

$$= |x - c|.$$

Let $\varepsilon > 0$ be given. Then $|g(x) - g(c)| < \varepsilon$ for $|x - c| < \delta$, $\delta = \varepsilon > 0$.

Hence $g$ is continuous for all $c \in \mathbf{R}$.

It is clear that $h = fog$, where $g(x) = \sin x, x \in [0, \pi]$ and $f(x) = \sqrt{x}, x \ge 0$. By Theorem 9, $h(x) = \sqrt{\sin x}$ is continuous on $[0, \pi]$.

**Example 56.** (*i*) *Show that the function* $h(x) = \sqrt{1 - x^2}$ *is continuous on* $[-1, 1]$.

(*ii*) *Prove that the function* $h(x) = \log \sin x$ *is continuous on* $[0, \pi]$.

(*iii*) *Prove that the function* $h(x) = e^{\sin x}$ *is continuous on* $\mathbf{R}$.

**Hint.** (*i*) Let $f(x) = 1 - x^2, x \in [-1, 1]$, $g(x) = \sqrt{x}, x \in [0, 1]$.

Then $h = gof$ is continuous on $[-1, 1]$

(*ii*) Let $f(x) = \sin x, x \in [0, \pi]$ and $g(x) = \log x, x > 0$.

Then $h = gof$ is continuous on $[0, \pi]$

(*iii*) $h = gof$, where $f(x) = \sin x, x \in \mathbf{R}$ and $g(x) = e^x, x \in \mathbf{R}$.

**Example 57.** *Given examples of two discontinuous functions f and g such that*

(*i*) *fog is continuous but gof is not continuous.* [D.U., B.C.A. 1998]

(*ii*) *fog and gof are both continuous.*

**Solution.** (*i*) We define two functions $f$ and $g$ as follow :

$$f(x)=\begin{cases}0, \text{ when } x\in \mathbf{Q}\\ x, \text{ when } x\in \mathbf{R}\sim\mathbf{Q}\end{cases}$$

and
$$g(x)=\begin{cases}1, \text{ when } x\in \mathbf{Q}\\ 0, \text{ when } x\in \mathbf{R}\sim\mathbf{Q}.\end{cases}$$

Then $f$ is discontinuous at each non-zero point of $\mathbf{R}$ and $g$ is discontinuous at each point of $\mathbf{R}$ [See Examples 47 (a) and 44]

If $x\in \mathbf{Q}$, then $(fog)(x)=f(g(x))=f(1)=0.$

If $x\in \mathbf{R}\sim\mathbf{Q}$, then $(fog)(x)=f(g(x))=f(0)=0.$

$\therefore$ $(fog)(x)=0\ \forall\ x\in \mathbf{R}$. Hence $fog$ is continuous $\forall\ x\in \mathbf{R}$.

If $x\in \mathbf{Q}$, then $(gof)(x)=g(f(x))=g(0)=1.$

If $x\in \mathbf{R}\sim\mathbf{Q}$, then $(gof)(x)=g(f(x))=g(x)=0.$

Thus
$$(gof)(x)=\begin{cases}1, \text{ when } x\in \mathbf{Q}\\ 0, \text{ when } x\in \mathbf{R}\sim\mathbf{Q}.\end{cases}$$

By Example 44, $gof$ is discontinuous $\forall\ x\in \mathbf{R}$.

(*ii*) We define two functions $f$ and $g$ on $\mathbf{R}$ as follow :

$$f(x)=\begin{cases}1, \text{ when } x\in \mathbf{Q}\\ -1, \text{ when } x\in \mathbf{R}\sim\mathbf{Q}\end{cases}$$

and
$$g(x)=\begin{cases}-1, \text{ when } x\in \mathbf{Q}\\ 1, \text{ when } x\in \mathbf{R}\sim\mathbf{Q}.\end{cases}$$

Thus $f$ and $g$ are discontinuous at all points of $\mathbf{R}$. [See Example 43]

If $x\in \mathbf{Q}$, then $(fog)(x)=f(g(x))=f(-1)=1.$ $(\because\ -1\in \mathbf{Q})$

If $x\in \mathbf{R}\sim\mathbf{Q}$, then $(fog)(x)=f(g(x))=f(1)=1.$ $(\because\ 1\in \mathbf{Q})$

$\Rightarrow$ $(fog)(x)=1\ \forall\ x\in \mathbf{R}$ $\Rightarrow$ $fog$ is a constant function on $\mathbf{R}$.

Hence $fog$ is continuous at all points of $\mathbf{R}$.

It may be noticed that $(gof)(x)=-1\ \forall\ x\in \mathbf{R}$ and so $gof$ is continuous at all points of $\mathbf{R}$.

## 5.8. Boundedness of Continuous Functions

**Definition.** *A function* $f:[a,b]\to \mathbf{R}$ *is said to be* **bounded** *in the interval* $[a,b]$, *if there exist two numbers k and K such that*

$$k\le f(x)\le K\ \ \forall\ \ x\in[a,b].$$

*Thus a function is* **bounded** *in* $[a,b]$, *if it is both bounded above and bounded below in* $[a,b]$.

**Illustrations**

1. $f(x)=x^2$ is bounded in $[1, 2]$, since $1\le f(x)\le 4\ \forall\ x\in[1,2]$.
2. $f(x)=1/x$ is bounded in $]0, 1[$.
3. $f(x)=1/(x-1)$ is not bounded in $[1, 4]$.

**Note.** The following results proved in earlier Chapters will be used in the proofs of some theorems :

(*a*) *Every bounded sequence has a limit point.*

(*b*) *A number p is a limit point of a sequence* $\langle a_n\rangle$ *iff there exists a subsequence* $\langle a_{n_k}\rangle$ *of* $\langle a_n\rangle$ *such that* $a_{n_k}\to p$.

(*c*) *Every closed interval* $[a, b]$ *is a closed set.*

(*d*) *A closed set contains all its limit points.*

(*e*) *If p is a limit point of a set S and* $S\subset T$, *then p is also a limit point of T.*

**Theorem 10.** *If a function f is continuous on a closed bounded interval* $[a,b]$, *then it is bounded in* $[a,b]$.

[D.U., Physics (H) 1997 ; Maths (H) 1996, 92 ;
B.Sc. (G) 1996, 94 ; B.A. (P) 1997, 90]

**Proof.** Let, if possible, $f$ be *not* bounded above in $[a, b]$.

Then for *each* positive integer $n$, we can find a point $x_n\in[a,b]$, such that

$$f(x_n)>n\ \ \forall\ \ n\in \mathbf{N}. \qquad \text{...(1)}$$

Since $x_n\in[a,b]$ for each $n\in\mathbf{N}$, $a\le x_n\le b\ \ \forall\ \ n\in\mathbf{N}$.

Thus $\langle x_n\rangle$ is a bounded sequence and so it must have a limit point say $p$ (Bolzano-Weierstrass Theorem).

Obviously, $p$ is a limit point of $[a, b]$. [Result (*e*)]

Now $[a, b]$ being a closed interval is a closed set and so $p\in[a,b]$.

Since $p$ is a limit point of the sequence $\langle x_n\rangle$, therefore, there exists a subsequence $\langle x_{n_k}\rangle$ of $\langle x_n\rangle$ such that $x_{n_k}\to p$. ...(2)

From (1), $f(x_{n_k})>n_k$ for all $k$ $\Rightarrow$ $<f(x_{n_k})$ diverges to $\infty$

$\Rightarrow f(x_{n_k})$ does not converge to $f(p)$. ...(3)

From (2) and (3), it follows that $f$ is not continuous at the point $p\in[a,b]$ (Theorem 8).

This is contrary to the hypothesis that $f$ is continuous at every point of $[a, b]$. Hence $f$ is bounded above on $[a, b]$.

Now we show that $f$ is bounded below on $[a, b]$.

Since $f$ is continuous in $[a, b]$, $-f$ is also continuous in $[a, b]$ and so $-f$ is bounded above (as proved above). Consequently, there exists some $k\in\mathbf{R}$ such that

$$-f(x)\le k\ \forall\ x\in[a,b]$$

$$\Rightarrow \qquad f(x)\ge -k \text{ or } -k\le f(x)\ \forall\ x\in[a,b]$$

$\Rightarrow f$ is bounded below in $[a, b]$.

**Hence** $f$ is bounded in $[a, b]$.

**Theorem 11.** *If a function $f$ is continuous an a closed bounded interval $[a, b]$, then it attains its bounds in $[a, b]$.*

**Proof.** Since $f$ is continuous in $[a, b]$, therefore, it is bounded in $[a, b]$. Let $M = \sup f$ and $m = \inf f$.

$$\therefore \qquad m \le f(x) \le M \quad \forall \; x \in [a, b].$$

We shall show that there exist points $\alpha, \beta \in [a, b]$ such that

$$f(\alpha) = M \quad \text{and} \quad f(\beta) = m.$$

We shall prove it by contradiction.

$$\text{Let} \qquad f(x) \ne M \quad \forall \; x \in [a, b]$$

$$\Rightarrow \qquad M - f(x) \ne 0, \quad \forall \; x \in [a, b].$$

Since $f$ is given to be continuous in $[a, b]$ and $M$, being a constant function, is also continuous in $[a, b]$, therefore $M - f(x)$ is continuous in $[a, b]$. Since $M - f(x) \ne 0$, $\forall \; x \in [a, b]$, therefore

$$\frac{1}{M - f(x)} \text{ is continuous in } [a, b].$$

$$\Rightarrow \qquad \frac{1}{M - f(x)} \text{ is bounded in } [a, b]. \qquad \text{(Theorem 10)}$$

$$\text{Let} \qquad K = \sup \frac{1}{M - f(x)} \quad \text{and} \quad k = \inf \frac{1}{M - f(x)}.$$

$$\Rightarrow \qquad \frac{1}{M - f(x)} \le K \quad \forall \; x \in [a, b] \quad \Rightarrow M - f(x) \ge \frac{1}{K} \quad \forall \; x \in [a, b]$$

$$\Rightarrow \qquad f(x) \le M - \frac{1}{K} \quad \forall \; x \in [a, b].$$

$\Rightarrow M - \frac{1}{K}$ $(< M)$ is an upper bound of $f$, which contradicts the fact that $M$ is the l.u.b. of $f$.

Hence there exists some $\alpha \in [a, b]$ such that $f(\alpha) = M$.

Similarly, there exists some $\beta \in [a, b]$ such that $f(\beta) = m$.

**Remark.** *If a function is not continuous on a closed interval, then it may not attain its bounds as shown below :*

**1.** The function $f(x) = x \; \forall \; x \in [0, 1[$ is continuous and bounded in $[0, 1[$, $0 \le f(x) < 1 \; \forall \; x \in [0, 1[$ ; $\inf f = 0$ and $\sup f = 1$. Here $\inf f$ is attained but $\sup f$ is not attained, since there is no point of $[0, 1[$ at which $f(x) = 1$.

(Observe that the domain of $f$ is not a closed interval)

**2.** The function $f(x) = x \; \forall \; x \in \,]0, 1]$ is continuous and bounded in $0, 1]$, attains the supremum 1 and does not attain the infimum 0.

**3.** The function $f(x) = x \; \forall \; x \in \,]0, 1[$ is continuous and bounded in $0, 1[$, but attains neither the supremum 1 nor the infimum 0.

**Ex. 1.** *Show that a continuous function on a closed bounded interval s bounded above. Give an example to show that the conclusion fails if the nterval is not closed.* [D.U., B.Sc. (G) 1996]

**Hint.** Prove Theorem 10. The function $f(x) = 1/x \; \forall \; x \in \,]0, 1]$ is continuous in $]0, 1]$, which is not bounded above. Notice that $]0, 1]$ is not a closed interval.

**Ex. 2.** *If $f$ is continuous on $[a, b]$, then there exists a real number $u$ such that*

$$f(x) \le u \quad \forall \quad x \in [a, b].$$

**Hint.** This is Theorem 10. Show that $f$ is bounded above.

**Ex. 3.** *Prove that every function $f$ defined and continuous in a closed interval is bounded above therein and also attains its supremum.*

[D.U., Maths (H) 1996]

**Hint.** Prove Theorems 10 and 11.

**Ex. 4.** *Let $f$ be continuous $[a, b]$. Prove that $f$ attains its supremum on $[a, b]$.* [D.U., B.Sc. (G) 1999]

**Hint.** Prove Theorem 11.

**Ex. 5.** *Prove that if $f$ is continuous in $[a, b]$ and $c$ is the infimum of $f$ in $[a, b]$, then there exists an $x_0$ in $[a, b]$ such that $f(x_0) = c$.*

[D.U., B.A. (P) 1995]

**Proof.** We have $c = \inf f$ in $[a, b]$ *i.e.*, $c \le f(x) \; \forall \; x \in [a, b]$.

Let, if possible, $f(x) \ne c \; \forall \; x \in [a, b]$. It follows that

$$\frac{1}{f(x) - c} \text{ is continuous in } [a, b]$$

$$\Rightarrow \frac{1}{f(x) - c} \text{ is bounded above in } [a, b] \qquad \text{(Theorem 10)}$$

$$\Rightarrow \frac{1}{f(x) - c} \le k \quad \forall \quad x \in [a, b], \; k \text{ being some real number}$$

$$\Rightarrow f(x) - c \ge \frac{1}{k} \; \forall \; x \in [a, b] \Rightarrow c + \frac{1}{k} \le f(x) \quad \forall \quad x \in [a, b]$$

$\Rightarrow c + (1/k)$ is a lower bound of $f$ in $[a, b]$, where $c + (1/k) > c$.

The contradicts the given condition $c = \inf f$. Hence there exists some $x_0 \in [a, b]$ such that $f(x_0) = c$.

**Ex. 6.** *Let $f$ be continuous on a closed interval $[a, b]$. Show that $\sup f$ and $\inf f$ are attained in $[a, b]$.* [D.U., B.Sc. (G) 1994]

**Theorem 12.** *If a function $f$ is continuous in $[a, b]$ and $c \in \,]a, b[$ such that $f(c) \ne 0$, then there exists some $\delta > 0$ such that $f(x)$ has the same sign as $f(c)$ for all $x \in \,]c - \delta, c + \delta[$.*

**Proof.** Since $f$ is continuous at $c$, for any $\varepsilon > 0$, $\exists$ some $\delta > 0$, such that

$$|f(x) - f(c)| < \varepsilon, \text{ when } \quad |x - c| < \delta.$$

$$\textit{i.e.,} \qquad f(c) - \varepsilon < f(x) < f(c) + \varepsilon, \text{ when } \quad c - \delta < x < c + \delta. \qquad \text{...(1)}$$

**Case I.** Let $f(c) > 0$. We choose $\varepsilon > 0$ such that

$$\varepsilon < f(c) \quad \Rightarrow \quad f(c) - \varepsilon > 0. \qquad \text{...(2)}$$

From (1) and (2), $f(x) > f(c) - \varepsilon > 0$, when $x \in \,]c - \delta, c + \delta[$.

**Case II.** Let $f(c) < 0 \Rightarrow -f(c) > 0$. We choose $\varepsilon > 0$ such that

$$\varepsilon < -f(c) \quad \Rightarrow \quad f(c) + \varepsilon < 0. \qquad \text{...(3)}$$

From (1) and (3), $f(x) < f(c) + \varepsilon < 0$, when $x \in \,]c - \delta, c + \delta[$.

Hence $f(x)$ has the same sign as $f(c)\ \forall\ x \in \,]c - \delta, c + \delta[$.

**Corollary.** (*i*) *If a function f is continuous at $x = a$ and $f(a) \neq 0$, then there exists some $\delta > 0$, such that $f(x)$ has the same sign as $f(a)$ $\forall$ $x \in [a, a + \delta[$.*

(*ii*) *If a function f is continuous at $x = b$ and $f(b) \neq 0$, then there exists some $\delta > 0$, such that $f(x)$ has the same sign as $f(b)$ $\forall\ x \in \,]b - \delta, b\,]$.*

**Ex. 7.** *Let f be continuous on $[a, b]$ and let $x_0$ be such that $a < x_0 < b$. If $f(x_0) < 0$, then show that $f(x) < 0\ \forall\ x \in \,]x_0 - \delta_0, x_0 + \delta_0[$ for some $\delta_0 > 0$.* [D.U., B.Sc. (G) 1993]

**Hint.** This is case II of Theorem 12.

**Ex. 8.** *Let f be continuous on $[a, b]$ and let $x_0$ be such that $a < x_0 < b$. If $f(x_0) > 0$, then show that $f(x) > 0\ \forall\ x \in \,]x_0 - \delta, x_0 + \delta[$ for some $\delta > 0$.*

**Hint.** This is case I of Theorem 12.

**Theorem 13.** *If a function f is continuous in $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, then there exists some point $c \in \,]a, b[$ such that*

$$f(c) = 0. \qquad \text{[D.U., Physics (H) 1994]}$$

**Proof.** Since $f(a)$ and $f(b)$ are of opposite signs, we may take

$$f(a) > 0 \quad \text{and} \quad f(b) < 0.$$

Let $S$ be a subset of $[a, b]$ defined as follows :

$$S = \{x : a \le x \le b \text{ and } f(x) > 0\}. \qquad \text{...(1)}$$

Then $S$ is non-empty $[\because f(a) > 0 \Rightarrow a \in S]$

nd $S$ is bounded above with $b$ as its upper bound.

By order-completeness property of **R**, $S$ has the supremum.

Let $c = \sup S$, $c \in [a, b]$.

Now we shall prove that $c \neq a$ and $c \neq b$, so that $c \in \,]a, b[$.

Since $f$ is continuous at $x = a$ and $f(a) > 0$, $\exists$ a $\delta_1 > 0$ such that

$f(x) > 0\ \ \forall\ \ x \in [a, a + \delta_1[$, by corollary of Theorem 12

$\Rightarrow\ [a, a + \delta_1[\, \subset S$, by (1).

If $c = a$, then $\sup S = a \Rightarrow$ every element of $S \le a$

$\Rightarrow a + \delta_1 \le a. \quad [\because [a, a + \delta_1] \subset S]$

This is impossible ($\because a + \delta_1 > a$) and so $c \neq a$.

Now we show that $c \neq b$. Since $f(b) < 0$, $\exists$ a $\delta_2 > 0$ such that

$f(x) < 0\ \ \forall\ \ x \in \,]b - \delta_2\, b]$, [by corollary of Theorem 12] ...(2)

$\Rightarrow b - \delta_2$ is an upper bound of $S$, for otherwise, there exists some $S$ such that $y > b - \delta_2$

*i.e.,* $\quad b - \delta_2 < y \le b \Rightarrow f(y) < 0$, by (2). ...(3)

As $y \in S, f(y) > 0$, by (1). This contradicts (3).

Since $c = \sup S$, so $c \le b - \delta_2 \Rightarrow c \neq b$.

Thus $c \neq a$ and $c \neq b \Rightarrow c \in \,]a, b[$.

Finally, we show that $f(c) = 0$.

**Case I.** Let $f(c) > 0$.

Since $f$ is continuous at $c$ and $f(c) > 0$, so $\exists$ a $\delta_3 > 0$ such that

$$f(x) > 0\ \ \forall\ \ x \in \,]c - \delta_3, c + \delta_3[. \qquad \text{...(4)}$$

Let $\alpha$ be a number such that $c < \alpha < c + \delta_3$. ...(5)

$\therefore \quad \alpha \in \,]c, c + \delta_3[\ \Rightarrow f(\alpha) > 0$, by (4) $\Rightarrow \alpha \in S$, by (1).

$\therefore \quad \alpha \le c \quad (\because c = \sup S)$ or $c \ge \alpha$, which contradicts (5).

Thus $f(c) \not> 0$.

**Case II.** Let $f(c) < 0$.

Since $f$ is continuous at $x = c$ and $f(c) < 0$, so $\exists$ a $\delta_4 > 0$ such that

$$f(x) < 0\ \ \forall\ x \in \,]c - \delta_4, c + \delta_4[. \qquad \text{...(6)}$$

Now $c = \sup S \Rightarrow \exists$ at least one $\beta \in S$ such that $c - \delta_4 < \beta \le c$

$\Rightarrow \beta \in \,]c - \delta_4, c] \ \Rightarrow f(\beta) < 0$, by (6).

But $\beta \in S \Rightarrow f(\beta) > 0$, which is a contradiction.

$\therefore \quad f(c) \not< 0$. Also $f(c) \not> 0$.

Hence $f(c) = 0$ for some $c \in \,]a, b[$.

**Remark.** The above theorem may be restated as follows :

*If a function f is continuous in $[a, b]$, and $f(a) > 0 > f(b)$, then there exists a point $c \in \,]a, b[$ such that $f(c) = 0$.*

*Or*

*If a function f is continuous in $[a, b]$ and $f(a)\,f(b) < 0$, then there exists a point $c \in \,]a, b[$ such that $f(c) = 0$.*

[D.U., Physics (H) 1996, 94 ; B.Sc. (G) 1994, 90 ; Maths (H) 1997]

Note that $f(a)\,f(b) < 0 \Rightarrow f(a)$ and $f(b)$ must be of opposite signs.

**Theorem 14. (Intermediate Value Theorem)**

*If a function f is continuous in $[a, b]$ and $f(a) \neq f(b)$, then f assumes every value between $f(a)$ and $f(b)$.* [D.U., B.A. (P) 1996 ; B.Sc. (G) 1997]

**Proof.** Suppose that $f(a) < f(b)$ and let $M$ be any number lying between $f(a)$ and $f(b)$, *i.e.,* $\quad f(a) < M < f(b)$. ...(1)

We have to show that $\exists$ some point $c \in [a, b]$ such that $f(c) = M$.

Let us define a function $\phi$ on $[a, b]$ as follows :

$$\phi(x) = f(x) - M, \quad \forall \quad x \in [a, b]. \qquad \text{...(2)}$$

Since $f$ is given to be continuous in $[a, b]$ and $M$ being a constant is always continuous, so $\phi$ is continuous in $[a, b]$.

From (2), $\phi(a) = f(a) - M$ and $\phi(b) = f(b) - M$.

By (1), $f(a) - M < 0$ and $f(b) - M > 0$

$\Rightarrow \phi(a)$ and $\phi(b)$ are of opposite signs,

By Theorem 13, there exists some point $c \in \,]a, b[$ such that

$$\phi(c) = 0 \Rightarrow f(c) - M = 0, \text{ by (2)}.$$

Hence $f(c) = M, c \in \,]a, b[$.

**Remark.** *If a function f is not continuous on a closed interval, then the conclusion of the Intermediate Value Theorem may not hold :*

Consider a function on [0, 1] as follows :

$$f(x) = x + 1 \quad \forall \quad x \in \,]0, 1],$$
$$f(0) = 0.$$

Obviously, $f$ is continuous on ]0, 1]. Further

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0} (x + 1) = 1. \text{ Also } f(0) = 0.$$

$\therefore \lim\limits_{x \to 0+} f(x) \neq f(0)$ and so $f$ is not continuous at 0.

*i.e.*, $f$ is not continuous in the closed interval [0, 1].

Now $f(0) = 0, f(1) = 2$.

We see that $f(0) < 1 < f(1)$. But there does not exist any $x \in \,]0, 1]$ such that $f(x) = 1$.

**Theorem 15.** *If a function f is continuous in a closed interval* [*a*, *b*], *then it assumes every value between its bounds.*

*Or*

*Let f be a continuous function on* [*a*, *b*] *and let* $M = \sup f$ *and* $m = \inf f$. *Show that f attains every value between m and M.*

[D.U., B.Sc. (G) 1998]

**Proof.** Since $f$ is continuous in the closed interval $[a, b]$, it is bounded a, b] and attains its bounds say $m$ and $M$, where

$$m = \inf f \text{ and } M = \sup f. \qquad \text{[Theorem 10]}$$

Further there exist points $\alpha, \beta \in [a, b]$ such that

$$f(\alpha) = m \text{ and } f(\beta) = M. \qquad \text{[Theorem 11]} \qquad ...(1)$$

We may assume that $\alpha < \beta$, so that $[\alpha, \beta] \subset [a, b]$.

Since $f$ is continuous in $[a, b]$, so $f$ is continuous in $[a, \beta]$. By Inter-diate Value Theorem, $f$ assumes every value between $f(\alpha)$ and $f(\beta)$. nce, by (1), $f$ assumes every value between its bounds $m$ and $M$.

**Theorem 16.** *The image of a closed interval under a continuous ction is a closed interval.*

**Proof.** Let $f : [a, b] \to \mathbf{R}$ be continuous in the closed interval $[a, b]$. e have to show that

$$f[a, b] = \{f(x) : a \leq x \leq b\} \qquad ...(1)$$

a closed interval.

Since $f$ is continuous in the closed interval $[a, b]$, it is bounded in $[a, b]$ and attains its bounds, say $m = \inf f$ and $M = \sup f$

*i.e.*, $$m \leq f(x) \leq M \quad \forall \quad x \in [a, b]. \qquad ...(2)$$

Thus there exist points $\alpha, \beta \in [a, b]$ such that

$$f(\alpha) = m \text{ and } f(\beta) = M.$$

We may assume that $\alpha < \beta$, so that $[\alpha, \beta] \subset [a, b]$.

Since $f$ is continuous in $[a, b]$, so $f$ is continuous in $[\alpha, \beta]$. By Intermediate Value Theorem, $f$ assumes every value between $f(\alpha)$ and $f(\beta)$ *i.e.*, $f$ assumes every value between $m$ and $M$. ...(3)

From (1) and (2), we get

$$f[a, b] \subset [m, M]. \qquad ...(4)$$

Conversely, let $x \in [m, M]$ so that $m \leq x \leq M$.

Using (3), there exists some $c \in [a, b]$ such that $f(c) = x$.

Using (1), $x \in f[a, b], \forall x \in [m, M]$.

$\therefore$ $[m, M] \subset f[a, b]$. ...(5)

From (4) and (5), $f[a, b] = [m, M]$, which is a closed interval. Hence the theorem.

**Theorem 17. (Inverse Function Theorem)**

*If a function f defined on the closed interval* [*a*, *b*] *is continuous on* [*a*, *b*] *and one-to-one, then* $f^{-1}$ *is also continuous.*

*Or*

*Let* $f : [a, b] \to \mathbf{R}$ *be continuous and one-to-one. Then* $f^{-1} : f([a, b]) \to [a, b]$ *is continuous.*

**The proof is omitted.**

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**Example 58.** *Show that if f and g are continuous on* [*a*, *b*] *and if* $f(a) < g(a)$ *and* $f(b) > g(b)$, *then there exists some* $c \in \,]a, b[$ *satisfying* $f(c) = g(c)$.

**Solution.** Let $h(x) = f(x) - g(x) \quad \forall \quad x \in [a, b]$. ...(1)

Since $f$ and $g$ are continuous on $[a, b]$, so by (1), $h$ is continuous on $[a, b]$.

Also $h(a) = f(a) - g(a) < 0, \qquad (\because f(a) < g(a))$

and $h(b) = f(b) - g(b) > 0. \qquad (\because f(b) > g(b))$

Thus $h$ is continuous on $[a, b]$ where $h(a), h(b)$ are of opposite signs.

Hence, by Theorem 13, there exists some $c \in \,]a, b[$ such that

$$h(c) = 0 \Rightarrow f(c) - g(c) = 0 \Rightarrow f(c) = g(c).$$

**Example 59.** *Let f be continuous on* [0, 1] *and let* $f(x)$ *be in* [0, 1] *for each x in* [0, 1]. *Prove that* $f(x) = x$ *for some x in* [0, 1].

[D.U., Maths (H) 1994]

**Solution.** We are given that $f(x) \in [0, 1] \; \forall \; x \in [0, 1]$ ...(1)

*i.e.*, $$0 \leq f(x) \leq 1 \quad \forall \quad [0, 1].$$

or
$$nm \le f(x_1) + f(x_2) + \dots + f(x_n) \le nM$$

or
$$m \le \frac{1}{n}[f(x_1) + f(x_2) + \dots + f(x_n)] \le M, \qquad \dots(1)$$

where $m$ and $M$ are bounds of $f$.

Since $f$ is continuous in $[a, b]$, it attains every value between its bounds [Theorem 15]. Using this property in (1), there exists some point $c \in [a, b]$ such that

$$f(c) = \frac{1}{n}[f(x_1) + f(x_2) + \dots + f(x_n)].$$

**Example 64.** *Let $f$ be continuous on $[a, b]$ and $x_1, x_2 \in [a, b]$. Show that there exists a point $c \in [a, b]$ such that*

$$f(c) = \frac{1}{2}[f(x_1) + f(x_2)]$$

**Hint.** Take $n = 2$ in Example 63.

**Example 65.** *Let*

$$f(x) = \lim_{n \to \infty} \frac{\log(2 + x) - x^{2n} \sin x}{1 + x^{2n}}, \; x \ge 0. \qquad \dots(1)$$

*Show that $f(0)$ and $f(\frac{1}{2}\pi)$ differ in sign. Why does $f(x)$ not vanish in $[0, \frac{1}{2}\pi]$ ? Explain.* [D.U., B.C.A. 1999]

**Solution.** We know that

$$x^{2n} \to 0 \quad \text{as} \quad n \to \infty, \quad \text{if} \quad 0 \le x < 1, \qquad \dots(2)$$

and
$$x^{2n} \to \infty, \text{as} \quad n \to \infty, \quad \text{if} \quad x > 1$$

*i.e.*,
$$x^{-2n} \to 0 \quad \text{as} \quad n \to \infty, \quad \text{if} \quad x > 1. \qquad \dots(3)$$

From (1) and (2), $f(x) = \log(2 + x)$, if $0 \le x < 1$. ...(4)

Also $f(1) = \frac{1}{2}(\log 3 - \sin 1)$, if $x = 1$. ...(5)

From (1) and (3), $f(x) = -\sin x$, if $x > 1$. ...(6)

$$\left[ \because f(x) = \lim_{n \to \infty} \frac{x^{-2n} \log(2 + x) - \sin x}{x^{-2n} + 1} = -\sin x, \text{ if } x > 1. \right]$$

From (4) and (6), $f(0) = \log 2 > 0$ and $f(\frac{1}{2}\pi) = -1 < 0$.

Thus $f(0)$ and $f(\frac{1}{2}\pi)$ differ in sign.

From (4), (5) and (6); it is clear that $f$ does not vanish in $[0, \frac{1}{2}\pi]$ *i.e.*, $f(x) \ne 0 \; \forall \; x \in [0, \frac{1}{2}\pi]$, even though $f(0)$ and $f(\frac{1}{2}\pi)$ are of opposite signs [see Theorem 13]

The reason for this is that $f$ is **not** continuous at $1 \in [0, \frac{1}{2}\pi]$ as shown below :

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1} \log(2 + x) = \log 3, \text{ by (4)}$$

(Notice that $\log x$ is continuous $\forall \; x > 0$)

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1} (-\sin x) = -\sin 1, \text{ by (6).}$$

(Notice that $\sin x$ is continuous $\forall\, x \in \mathbf{R}$)

Thus $f$ is not continuous at $x = 1$. Consequently, $f$ is not continuous in the closed interval $[0, \frac{1}{2}\pi]$.

**Remark.** The above example shows that the hypothesis as well as the conclusion of the Intermediate value Theorem are not satisfied for the function $f$ in $[0, \frac{1}{2}\pi]$.

**Example 66.** *Show that the function*

$$\phi(x) = \lim_{n \to \infty} \frac{x^{2n+2} - \cos x}{x^{2n} + 1}$$

*does not vanish anywhere in the interval* $[0, 2]$, *though* $\phi(0)$ *and* $\phi(2)$ *differ in sign.*

**Hint.** We have $\phi(x) = -\cos x$ for $0 \le x < 1$ ;

$$\phi(1) = \frac{1}{2}(1 - \cos x) \text{ and } \phi(x) = x^2 \text{ for } x > 1.$$

$$\phi(0) = -1 < 0,\ \phi(2) = 4 > 0.$$

Verify that $\phi$ is discontinuous at $x = 1$.

**Example 67.** *Give an example of a function which satisfies the conclusion but not the hypothesis of the Intermediate value theorem.*

**Solution.** We define a function $f$ on $[0, 1]$ as follows :

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases}$$

Then $f$ is not continuous in $[0, 1]$ except at $x = \frac{1}{2}$ [see Example 49].

However, $f$ takes every value between 0 and 1.

**Example 68.** *Show that the function* $x \to x^{1/n}$, *defined for all* $x \ge 0$ *and* $n \in \mathbf{N}$, *is continuous.*

**Solution.** Consider the function

$$f : [0, b] \to [0, b^n] \text{ given by } f(x) = x^n \ (b \in \mathbf{R}).$$

Then $f$ is continuous and one-to-one on $[0, b]$. By Theorem 17, $f^{-1}$ is continuous on $[0, b^n]$, where $f^{-1}(y) = x \Leftrightarrow f(x) = y \Leftrightarrow x^n = y \Leftrightarrow x = y^{1/n}$.

Thus $f^{-1}(y) = y^{1/n}$ or $f^{-1}(x) = x^{1/n}$ is continuous on $[0, b^n]$. To show that the root function is continuous for all $x \ge 0$, let $a \ge 0$ be arbitrary. We can choose some $b > 0$ such that $b^n > a$ *i.e.*, $a \in [0, b^n]$.

Hence $x \to x^{1/n}$ is continuous for all $a \ge 0$.

## 5.9 Uniform Continuity

**Definition.** *A function f defined on an interval I is said to be* **uniformly continuous in the interval** *I, if for each* $\varepsilon > 0$, *there exists some* $\delta > 0$ *such that*

$$|f(x_2) - f(x_1)| < \varepsilon, \text{ when } |x_2 - x_1| < \delta \text{ and for all } x_1, x_2 \in I.$$

**Remarks.**

**1.** It may be noted that whereas continuity of a function is defined *at a point*, the uniform continuity of a function is defined *in an interval*. Even when we say that a function $f$ is continuous in an interval it means that $f$ is continuous at *all points* of the interval.

**2.** In case of continuity of a function at a point $c$, *the choice of* $\delta > 0$ *depends upon* $\varepsilon > 0$ *and the point* $c$. But in case of uniform continuity of a function in an interval, *the choice of* $\delta > 0$ *depends only on* $\varepsilon > 0$ and not on a pair of points of the given interval.

**3.** *A function f is not* **uniformly continuous** *in an interval I, if there exists* **some** $\varepsilon > 0$, *such that* **for any** $\delta > 0$, *there exists a pair of points* $x, y \in I$ *for which* $|f(x) - f(y)| \ge \varepsilon$, *when* $|x - y| < \delta$.

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**Example 69.** *Show that the function defined by* $f(x) = x^2$ *is uniformly continuous in* $]-2, 2[$.

**Solution.** Let $\varepsilon > 0$ be any number and $x_1, x_2 \in\ ]-2, 2[$. Then

$$\begin{aligned} |f(x_2) - f(x_1)| &= |x_2^2 - x_1^2| = |(x_2 - x_1)(x_2 + x_1)| \\ &\le |x_2 - x_1|(|x_2| + |x_1|) \\ &= 4|x_2 - x_1| \quad (\because x_1 < 2, x_2 < 2) \end{aligned}$$

$$\therefore \quad |f(x_2) - f(x_1)| < \varepsilon, \text{ when } |x_2 - x_1| < \varepsilon/4$$

$$\text{or} \quad |f(x_2) - f(x_1)| < \varepsilon, \text{ when } |x_2 - x_1| < \delta\ (= \varepsilon/4)$$

$$\forall\, x_1, x_2 \in\ ]-2, 2[.$$

Hence the function $f$ is uniformly continuous in $]-2, 2[$.

**Example 70.** *Show that* $f(x) = x^2$ *is uniformly continuous in* $[-1, 1]$.

**Hint.** $x_1 \le 1, x_2 \le 1 \Rightarrow x_1 + x_2 \le 2$. $\delta = \varepsilon/2$.

**Example 71.** *Is the function* $f(x) = \frac{x}{x+1}$ *uniformly continuous for* $x \in [0, 2]$ ? *Justify your answer.* [D.U., Maths (H) 1995]

**Solution.** Let $x, y$ be two arbitrary points in $[0, 2]$. Then $x \ge 0$, $y \ge 0$ $\Rightarrow x + 1 \ge 1$ and $y + 1 \ge 1 \Rightarrow (x + 1)(y + 1) \ge 1$. ...(1)

$$\text{Now } |f(x) - f(y)| = \left| \frac{x}{x+1} - \frac{y}{y+1} \right| = \frac{|x - y|}{(x+1)(y+1)} \le |x - y|, \text{ by (1)}$$

Let $\varepsilon > 0$ be given. Taking $\delta = \varepsilon$, we see that

$$|f(x) - f(y)| < \varepsilon, \text{ whenever } |x - y| < \delta, \ \forall\, x, y \in [0, 2].$$

Hence $f$ is uniformly continuous in $[0, 2]$.

**Example 72.** *Show that the function f defined by* $f(x) = x^3$ *is uniformly continuous in the interval* $[0, 3]$.

**Solution.** Let $\varepsilon > 0$ and $x_1, x_2 \in [0, 3]$. $\therefore$ $x_1 \le 3, x_2 \le 3$.

**Solution.** Let $\varepsilon = \frac{1}{2}$ and $\delta$ be any positive number. We

positive integer $n$ such that

$$n > \frac{1}{\delta}. \qquad \ldots(1)$$

Let $x_1 = \frac{1}{n}$ and $x_2 = \frac{1}{n+1} \in\, ]0, 1]$. Then

$$|f(x_2) - f(x_1)| = \left| \frac{1}{x_2} - \frac{1}{x_1} \right| = |n + 1 - n| = 1 > \varepsilon,$$

and
$$|x_2 - x_1| = \left| \frac{1}{n+1} - \frac{1}{n} \right| = \frac{1}{n(n+1)} < \frac{1}{n} < \delta, \text{ by (1)}$$

$$\therefore \quad |f(x_2) - f(x_1)| > \varepsilon, \text{ when } |x_2 - x_1| < \delta.$$

Hence $1/x$ is not uniformly continuous on $]0, 1]$.

**Example 76.** *Give an example of a continuous function which is not uniformly continuous.* [D.U., Maths (H) 1997 ; B.Sc. (G) 1996]

**Solution.** By Example 75, $f(x) = 1/x$ is not uniformly continuous in $]0, 1]$. However, $f$ is continuous in $]0, 1]$, as $x \neq 0 \in\, ]0, 1]$ is continuous in $]0, 1]$.

**Example 77.** *Show that $f(x) = 1/x$ is not uniformly continuous on $]0, 1[$, but it is uniformly continuous on $[a, \infty[$, where $a > 0$.*

**Solution.** By Example 75, $f(x) = 1/x$ is not uniformly continuous on $]0, 1[$. Let $x_1, x_2 \in [a, \infty[$ so that $x_1 \geq a > 0$ and $x_2 \geq a > 0$. Let $\varepsilon > 0$ be given. Then

$$|f(x_1) - f(x_2)| = \left| \frac{1}{x_1} - \frac{1}{x_2} \right| = \frac{|x_1 - x_2|}{x_1 x_2} \leq \frac{|x_1 - x_2|}{a^2}.$$

$$\therefore \; |f(x_1) - f(x_2)| < \varepsilon, \text{ if } |x_1 - x_2| < \delta \; (\delta = \varepsilon\, a^2 > 0) \; \forall \; x_1, x_2 \in [a, \infty[.$$

Hence $f(x) = 1/x$ is uniformly continuous on $[a, \infty[$, $a > 0$.

**Example 78.** *Show that $f(x) = 1/x^2$ is uniformly continuous on $[a, \infty[$, $a > 0$.* [D.U., B.Sc. (G) 1994]

**Solution.** For $x, y \geq a > 0$ ; we see that

$$|f(x) - f(y)| = \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{1}{x} - \frac{1}{y} \right| \left| \frac{1}{x} + \frac{1}{y} \right|$$

or
$$|f(x) - f(y)| = \left| \frac{y-x}{xy} \right| \left( \frac{1}{x} + \frac{1}{y} \right) \leq \frac{2}{a} \frac{|x-y|}{|xy|} \qquad [\because\; x, y \geq a > 0]$$

$$\leq \frac{2}{a^3} |x - y| \left[ \because\; x \geq a, y \geq a \Rightarrow xy \geq a^2 \Rightarrow \frac{1}{xy} \leq \frac{1}{a^2} \right]$$

Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon a^3/2$. Then

$$|f(x) - f(y)| < \varepsilon, \quad \text{when} \quad |x - y| < \delta \ \forall \ x, y \geq a.$$

Hence $f$ is uniformly continuous on $[a, \infty]$.

**Example 79.** *Show that the function $f(x) = 1/x^2$ is uniformly continuous on $[a, \infty)$, where $a > 0$, but not uniformly continuous on $(0, \infty)$.*

[D.U., Physics (H) 1995]

**Solution.** By Example 78, $f(x) = 1/x^2$ is uniformly continuous in $[a, \infty)$, $a > 0$. Now we show that $f$ is not uniformly continuous in $(0, \infty)$. Let $\varepsilon = \frac{1}{2}$ and $\delta$ be any positive number. We can always choose a positive integer $n$ such that

$$n > \frac{1}{2\delta} \quad \text{or} \quad \frac{1}{2n} < \delta. \qquad \text{...(1)}$$

Let $\qquad x_1 = \frac{1}{\sqrt{n}}$ and $x_2 = \frac{1}{\sqrt{n+1}} \in (0, \infty)$. Then

$$|f(x_1) - f(x_2)| = \left| \frac{1}{x_1^2} - \frac{1}{x_2^2} \right| = |n - (n+1)| = 1 > \varepsilon,$$

and

$$|x_1 - x_2| = \left| \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right| = \frac{|\sqrt{n+1} - \sqrt{n}|}{\sqrt{n}\sqrt{n+1}}$$

$$= \frac{1}{\sqrt{n}\sqrt{n+1}\,(\sqrt{n+1} + \sqrt{n})}, \text{ on rationalising}$$

$$< \frac{1}{\sqrt{n}\,.\,2\sqrt{n}} \quad [\because \sqrt{n}\sqrt{n+1} > \sqrt{n} \text{ and } \sqrt{n+1} + \sqrt{n} > 2\sqrt{n}]$$

$$= \frac{1}{2n} < \delta, \text{ by (1)}.$$

Thus $\quad |f(x_1) - f(x_2)| > \varepsilon$, when $|x_1 - x_2| < \delta$.

Hence $f$ is not uniformly continuous on $(0, \infty)$.

**Example 80.** *Show that $f(x) = \sqrt{x}$ is uniformly continuous in* [0, 1].

**Solution.** Let $x, y \in [0, 1]$, where $x > y \geq 0$. Then

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \left| \frac{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})}{\sqrt{x} + \sqrt{y}} \right|$$

$$= \frac{x - y}{\sqrt{x} + \sqrt{y}} \leq \frac{x - y}{\sqrt{x}} \leq \frac{x - y}{\sqrt{x - y}} = \sqrt{x - y}.$$

$\therefore \qquad |f(x) - f(y)| \leq \sqrt{x - y}$.

Let $\varepsilon > 0$ be given. Then

$$|f(x) - f(y)| < \varepsilon, \quad \text{when} \quad \sqrt{x - y} < \varepsilon$$

or

$$|f(x) - f(y)| < \varepsilon, \quad \text{when} \quad |x - y| < \delta \,(= \varepsilon^2) \ \forall \ x, y \in [0, 1].$$

Hence $f$ is uniformly continuous in [0, 1].

**Example 81.** *Show that $\sin x$ is uniformly continuous on* $[0, \infty[$.

[D.U., B.A. (P) 1997]

**Solution.** Let $\varepsilon > 0$ be given and $x, y$ be any two points in $[0, \infty[$. Let $f(x) = \sin x$. Then

$$|f(x) - f(y)| = |\sin x - \sin y|$$

$$= \left| 2 \sin \frac{x - y}{2} \cos \frac{x + y}{2} \right|$$

$$= 2 \left| \sin \frac{x - y}{2} \right| \left| \cos \frac{x + y}{2} \right|$$

$$\leq 2 \left| \frac{x - y}{2} \right| .1 \qquad [\because |\sin \theta| \leq |\theta|, |\cos \theta| \leq. \ \forall \ \theta]$$

$\therefore \quad |f(x) - f(y)| \leq |x - y|$

$\Rightarrow \quad |f(x) - f(y)| < \varepsilon$, when $|x - y| < \delta$, $(\delta = \varepsilon) \ \forall \ x, y \in [0, \infty[$.

Hence $f(x) = \sin x$ is uniformly continuous on $[0, \infty[$.

**Example 82.** *Prove that $f(x) = \sin x^2$ is not uniformly continuous on* $[0, \infty[$.

[D.U., Physics (H) 1999 ; B.Sc. (G) 1997]

**Solution.** Let $\varepsilon = 1/2$ and $\delta$ be any positive number. We can choose a positive integer $n$ such that

$$n > \pi/\delta^2. \qquad \text{...(1)}$$

Let $\qquad x_1 = \sqrt{\frac{n\pi}{2}}$, $x_2 = \sqrt{(n+1)\frac{\pi}{2}} \in [0, \infty[$. Then

$$|f(x_2) - f(x_1)| = |\sin x_2^2 - \sin x_1^2|$$

$$= \left| \sin (n+1)\frac{\pi}{2} - \sin \frac{n\pi}{2} \right|$$

$$= \begin{cases} |0 - (\pm 1)| = 1, \text{ if } n \text{ is odd,} \\ |\pm 1 - 0| = 1, \text{ if } n \text{ is even.} \end{cases}$$

$\therefore \qquad |f(x_2) - f(x_1)| = 1 > \varepsilon,$

and

$$|x_2 - x_1| = \left| \frac{x_2^2 - x_1^2}{x_2 + x_1} \right| = \frac{\pi/2}{\sqrt{(n+1)\frac{\pi}{2}} + \sqrt{\frac{n\pi}{2}}},$$

$$< \frac{\pi}{2\left\{2\sqrt{\frac{n\pi}{2}}\right\}} < \frac{\pi}{\sqrt{n\pi}} = \sqrt{\frac{\pi}{n}} < \delta, \text{ by (1)}.$$

$\therefore \qquad |f(x_2) - f(x_1)| > \varepsilon$, when $|x_2 - x_1| < \delta$.

Hence $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty[$.

**Example 83.** *Show that $f(x) = \sin \frac{1}{x}$ is not uniformly continuous on* $]0, \infty[$.

**Solution.** Let $\varepsilon = \frac{1}{2}$ and $\delta$ be any positive number. We can choose a positive integer $n$ such that $n\pi > 1/\delta$. ...(1)

Let $x_1 = \dfrac{1}{n\pi}$ and $x_2 = \dfrac{2}{(2n+1)\pi} \in ]0, \infty[$.

Then $|f(x_1) - f(x_2)| = \left| \sin n\pi - \sin (2n+1)\dfrac{\pi}{2} \right| = 1 > \varepsilon$,

and $|x_1 - x_2| = \left| \dfrac{1}{n\pi} - \dfrac{2}{(2n+1)\pi} \right| = \dfrac{1}{n(2n+1)\pi} < \dfrac{1}{n\pi} < \delta$, by (1).

$\therefore \quad |f(x_1) - f(x_2)| > \varepsilon$, when $|x_1 - x_2| < \delta$.

Hence $f(x) = \sin \frac{1}{x}$ is not uniformly continuous on $]0, \infty[$.

## 5.10 Theorems on Uniform Continuity

**Theorem 18.** *Show that every uniformly continuous function on an interval is continuous on that interval, but the converse is not true.*

[D.U., B.Sc. (G) 1998, 95, 92 ; B.A. (P) 1995, 94 ;
Physics (H) 1995 ; Maths (H) 1998]

**Proof.** Let a function $f$ be uniformly continuous on an interval $I$. Then for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \text{ when } |x - y| < \delta \ \forall \ x, y \in I. \qquad ...(1)$$

Let $c$ be any point of $I$. Taking $y = c$ in (1), we obtain

$$|f(x) - f(c)| < \varepsilon, \text{ when } |x - c| < \delta.$$

Thus $f$ is continuous at the point $c$.

Since $c \in I$ is arbitrary, it follows that $f$ is continuous at each point of $I$. Hence $f$ is continuous on the interval $I$.

However, the converse of the theorem is not true.

A function which is continuous on an interval may not be uniformly continuous on that interval.

For example, the function $f(x) = 1/x \ \forall \ x \in ]0, 1]$ is continuous on $]0, 1]$, but $f$ is not uniformly continuous on $]0, 1]$. (See Example 75).

However, if a function is continuous on a *closed interval*, then it is necessarily uniformly continuous on that closed interval as proved in the following :

**Theorem 19.** *If a function f is continuous on a closed and bounded interval* $[a, b]$, *then it is uniformly continuous on* $[a, b]$.

[D.U., Maths (H) 1999, 96, 94 ; Physics (H) 1998]

**Proof.** Let, if possible, $f$ be not uniformly continuous on $[a, b]$. Then there exists *some* $\varepsilon > 0$ such that for *any* $\delta_n = 1/n \ (n \in \mathbf{N})$, *there is a pair of elements* $x_n, y_n \in [a, b]$ for which

$$|f(x_n) - f(y_n)| > \varepsilon \quad \text{when} \quad |x_n - y_n| < 1 \qquad ...(1)$$

Since $a \le x_n \le b$, $a \le y_n \le b$ for each $n$, the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ are bounded and so by Bolzano-Weierstrass theorem, they have limit points. Let $\alpha$ and $\beta$ be limit points of $\langle x_n \rangle$, $\langle y_n \rangle$, respectively. It follows that $\alpha$ and $\beta$ are limit points of $[a, b]$. Since a closed interval is a closed set and further a closed set contains all its limit points, so $\alpha, \beta \in [a, b]$.

Now $\alpha$ is a limit point of $\langle x_n \rangle$ implies that there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $\quad x_{n_k} \to \alpha$. ...(2)

Similarly, $y_{n_k} \to \beta$, where $\langle y_{n_k} \rangle$ is a subsequence of $\langle y_n \rangle$. ...(3)

From (1), we conclude that

$$|f(x_{n_k}) - f(y_{n_k})| > \varepsilon, \quad \text{when} \quad |x_{n_k} - y_{n_k}| < \frac{1}{n_k}.$$

It is clear that $x_{n_k} - y_{n_k} \to 0$ as $k \to \infty$

or $\qquad \lim x_{n_k} = \lim y_{n_k} \quad \Rightarrow \quad \alpha = \beta$, by (2) and (3).

But $\quad |f(x_{n_k}) - f(y_{n_k})| > \varepsilon \quad \Rightarrow \quad \lim f(x_{n_k}) \neq \lim f(n_{n_k})$,

provided the two limits exist.

In particular, $\quad \lim f(x_{n_k}) \neq f(\alpha)$ and $\lim (y_{n_k}) \neq f(\alpha)$.

Thus we have two sequences $\langle x_{n_k} \rangle$, $\langle y_{n_k} \rangle$ satisfying

$$x_{n_k} \to \alpha \quad \text{and} \quad y_{n_k} \to \alpha,$$

but $\langle f(x_{n_k}) \rangle$ and $\langle f(y_{n_k}) \rangle$ do not converge to $f(\alpha)$, $\alpha \in [a, b]$.

It follows that $f$ is not continuous at $\alpha \in [a, b]$, which is a contradiction to the given hypothesis. Hence $f$ must be uniformly continuous in $[a, b]$.

**Ex.** *Define uniform continuity of a function. Prove that a function defined in a closed interval is continuous if and only if it is uniformly continuous therein.* [D.U., Maths (H) 1988]

**Hint.** Prove Theorems 18 and 19.

### EXAMPLES

**Example 84.** *Show that the function* $f(x) = x^2$ *in* $[1, 2]$ *is uniformly continuous in* $[1, 2]$.

**Solution.** It is clear that the function $f(x) = x^2 \ \forall \ x \in [1, 2]$ is continuous in the closed interval $[1, 2]$. Hence $f$ is uniformly continuous in $[1, 2]$ (Theorem 19).

**Example 85.** *If f and g are uniformly continuous on an interval I, then prove that f + g is uniformly continuous on I.*

**Solution.** Since $f$ and $g$ are uniformly continuous on $I$, for any $\varepsilon > 0$, $\exists$ some $\delta_1 > 0$ and $\delta_2 > 0$ such that $\forall \ x, y \in I$,

$$|f(x) - f(y)| < \varepsilon/2, \quad \text{when} \quad |x - y| < \delta_1, \qquad ...(1)$$

$$|g(x) - g(y)| < \varepsilon/2, \quad \text{when} \quad |x - y| < \delta_2. \qquad ...(2)$$

...(3)

Let $\qquad \delta = \min(\delta_1, \delta_2)$.

Thus $f(x) = x + 1$ is uniformly continuous on **R**

Similarly, $g(x) = x - 1$ is uniformly continuous on **R**.

We shall now prove that $f(x)\, g(x) = x^2 - 1$ is not uniformly continuous on **R**. For any $\delta > 0$, let

$$x_1 = \frac{1}{\delta} \quad \text{and} \quad x_2 = \frac{1}{\delta} + \frac{\delta}{2}. \quad \text{Then} \quad |x_1 - x_2| = \frac{\delta}{2} < \delta \text{ and}$$

$$|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = \left| \frac{1}{\delta^2} - \left( \frac{1}{\delta} + \frac{\delta}{2} \right)^2 \right| = 1 + \frac{\delta^2}{4} > 1.$$

If we choose $\varepsilon = 1$, then $|f(x_1) - f(x_2)| \nless \varepsilon$ for $|x_1 - x_2| < \delta$. Hence the product $x^2 - 1$ of two uniformly continuous functions $x + 1$ and $x - 1$ on **R** is not uniformly continuous on **R**.

## EXERCISES

1. Show that $f(x) = 3x + 5$ is uniformly continuous on **R**.
2. Show that $f(x) = x^2$ is uniformly continuous in $[1, 2]$.
   [**Hint.** Similar to Example 69.]
3. Show that $f(x) = \sqrt{x}$ is uniformly continuous on $[1, 3]$.
   [**Hint.** Similar to Example 80.]
4. Show that $f(x) = 1/x$ is uniformly continuous in $]1, 3[$.
   [**Hint.** Similar to Example 77.]
5. Show that $f(x) = x^3$ is uniformly continuous in $[0, 2]$.
   [**Hint.** Similar to Example 72. $\delta = \varepsilon/12$.]
6. Show that $f(x) = x^2$ is not uniformly continuous on **R**.
   [**Hint.** Refer to Example 88. Let $x_1 = 1/\delta$, $x_2 = 1/\delta + \delta/2$, $\delta > 0$.]
7. Show that $f(x) = \dfrac{x}{1 + x^2}$ is uniformly continuous on **R**.
8. Show that $f(x) = \dfrac{2x}{2x - 1}$ is uniformly continuous on $[1, \infty[$.

   [**Hint.** $|f(x) - f(y)| = \dfrac{2|x - y|}{(2x - 1)(2y - 1)}$ $\forall\, x, y \in [1, \infty[$

   Since $x > 1$ and $y > 1$, $2x - 1 > 1$, $2y - 1 > 1$ and so
   $|f(x) - f(y)| < 2|x - y|$. For any $\varepsilon > 0$, choose $\delta = \frac{1}{2}\varepsilon$.]
9. If $f$ and $g$ are uniformly continuous on an interval $I$, then show that $f - g$ is uniformly continuous on $I$.
   [**Hint.** Proceed like Example 85.]
10. Prove that if $f$ is uniformly continuous on an interval $I$ and $k \in$ **R**, then $kf$ is uniformly continuous on $I$.

# 6

# Derivatives

## 6.1 Derivability of a Function

Let $f$ be a real valued function defined on an interval $[a, b]$ *i.e.*,

$$f : [a, b] \to \mathbf{R}.$$

Let $\quad a < c < b.$

**Definition 1.** *The* **left hand derivative** *of a function f at $x = c$, denoted by $Lf'(c)$ or $f'(c - 0)$ is defined as*

$$Lf'(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c}, \text{ provided the limit exists.}$$

**Definition 2.** *The* **right hand derivative** *of a function f at $x = c$, denoted by $Rf'(c)$ or $f'(c + 0)$, is defined as*

$$Rf'(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c}, \text{ provided the limit exists.}$$

**Definition 3.** *If $Lf'(c) = Rf'(c)$, we say that the function f is* **derivable** *at $x = c$. The common limit is called the* **derivative** *of f at $x = c$ and is denoted by $f'(c)$. Thus*

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

**Definition 4.** *If $Lf'(c) \neq Rf'(c)$, we say that the function f is* **not derivable** *at $x = c$.*

**Definition 5.** *The derivative of a function f at the* **left end point** *a, is defined as*

$$f'(a) = \lim_{x \to a+} \frac{f(x) - f(a)}{x - a}, \text{ provided the limit exists.}$$

**Definition 6.** *The derivative of a function f at the* **right end point** *b,* *s defined as*

$$f'(b) = \lim_{x \to b-} \frac{f(x) - f(b)}{x - b}, \text{ provided the limit exists.}$$

**Definition 7.** *A function f defined on an interval $[a, b]$ is said to be* **erivable in the interval** *$[a, b]$, if f is derivable at all points of the interval* *a, b].*

**Note.** *It may be noted that the function f will not be derivable in* *a, b], if it fails to be derivable even at one point of the interval $[a, b]$.*

We give the relationship between the derivability and continuity of a unction in the following :

**Theorem 1.** *Prove that if a function f is derivable at a point, it is continuous at that point. The converse need not be true.*

[D.U., Maths (H) 1996 ; B.Sc. (G) 1991]

**Proof.** Let $f$ be derivable at $x = c$, so that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Now $\lim\limits_{x \to c} [f(x) - f(c)] = \lim\limits_{x \to c} \left[ \frac{f(x) - f(c)}{x - c} \times (x - c) \right]$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c)$$

$$= f'(c) \times 0 = 0.$$

$\therefore \lim\limits_{x \to c} f(x) = f(c)$. Hence $f$ is continuous at $x = c$.

(*ii*) The converse may not be true. We shall prove that *The function $f(x) = |x|$ is continuous at $x = 0$ but not derivable at* $x = 0$.

We know $\quad |x| = x$ if $x \geq 0$,

$\quad = -x$ if $x < 0$.

Now $\quad \lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0} (-x) = 0,$

$\quad \lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0} (-x) = 0$. Also $f(0) = 0$.

$\therefore \quad \lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0+} f(x) = f(0).$

Hence $f$ is continuous at $x = 0$.

Now $\quad \dfrac{f(x) - f(0)}{x - 0} = \dfrac{|x| - 0}{x} = \dfrac{|x|}{x}.$

$$\therefore \quad Lf'(0) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(-x)}{x} = -1,$$

$$Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x}{x} = 1.$$

Thus $Lf'(0) \neq Rf'(0)$ implies $f$ is not derivable at $x = 0$.

## EXAMPLES

**Example 1.** *Show that the function*

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x = 0 \\ 0, & \text{if } x = 0 \end{cases}$$

*is continuous at $x = 0$ but not derivable at $x = 0$.*

**Solution.** We have $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \left( x \sin \frac{1}{x} \right) = 0.$

(See Example 2, Chapter 5)

Thus $\lim\limits_{x \to 0} f(x) = f(0)$ and so $f$ is continuous at $x = 0$.

Now $\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \sin \frac{1}{x}$, which does not exist.

(See Example 3, Chapter 5)

Hence $f$ is not derivable at $x = 0$.

**Example 2.** *Show that* $f(x) = |x| + |x - 1|$ *is continuous but not derivable at* $x = 0, 1$. [D.U., Physics (H) 1994]

**Solution.** Refer to Example 24, Chapter 5. We have seen that $f$ is continuous at $x = 0, 1$. Now we show that $f$ is not derivable at $x = 0$ and 1. We know

$$f(x) = 1 - 2x, \text{ when } x < 0,$$
$$f(x) = 1 \quad , \text{ when } 0 \le x < 1,$$
$$f(x) = 2x - 1, \text{ when } x \ge 1.$$

Now $\quad Lf'(0) = \lim\limits_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{(1 - 2x) - 1}{x} = -2,$

$$Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \frac{1 - 1}{x} = 0,$$

$\therefore \quad Lf'(0) \neq Rf'(0) \Rightarrow f$ is not derivable at $x = 0$.

Again $\quad Lf'(1) = \lim\limits_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \frac{1 - 1}{x - 1} = 0,$

$$Rf'(1) = \lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} = \frac{(2x - 1) - 1}{x - 1} = 2,$$

$\therefore \quad Lf'(1) \neq Rf'(1) \Rightarrow f$ is not derivable at $x = 1$.

**Example 3.** *Determine whether f is continuous and has derivative at e origin where*

$$f(x) = 2 + x \text{ if } x \ge 0, f(x) = 2 - x \text{ if } x < 0.$$

**Solution.** $\quad \lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0} (2 - x) = 2,$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0} (2 + x) = 2.$$

Also $\quad f(0) = 2 + 0 = 2.$

Hence $f$ is continuous at $x = 0$.

Now $\quad Lf'(0) = \lim\limits_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{(2 - x) - 2}{x} = -1,$

$$Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(2 + x) - 2}{x} = 1.$$

$\therefore \quad Lf'(0) \neq Rf'(0)$ and so $f$ is not derivable at $x = 0$.

**Example 4.** *Let* $f(x) = |x| - 1$ *for all* $x \in \mathbf{R}$. *Show that f is derivable at all points except* $x = 0$. *Also, show that* $Rf'(0) = 1$, $Lf'(0) = -1$.

**Solution.** We know $|x| = x$, if $x \ge 0$ and $|x| = -x$, if $x < 0$.

$$Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(x - 1) - (-1)}{x} = 1,$$

$$Lf'(0) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(-x - 1) - (-1)}{x} = -1$$

Thus $f$ is not derivable at $x = 0$. We have

$$f(x) = x - 1, \text{ if } x \ge 0 \text{ and } f(x) = -x - 1, \text{ if } x < 0.$$

$\therefore \quad f'(x) = 1 \quad$, if $x > 0$ and $f'(x) = -1 \quad$, if $x < 0$.

Hence $f$ is derivable at all points except $x = 0$.

**Example 5.** *Show that the function* $f(x) = x|x|$ *is derivable at the origin.*

**Solution.** We have

$$Lf'(0) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x(-x)}{x} = 0,$$

$$Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x(x)}{x} = 0.$$

$\therefore \quad Lf'(0) = Rf'(0) = 0.$

Hence $\quad f'(0) = 0.$

**Example 6.** *Let f be defined by setting*

$$f(x) = \begin{cases} x, & \text{if } x \le 0 \\ 0, & \text{if } x > 0. \end{cases}$$

*Show that f is derivable at all points except* $x = 0$.

**Solution.** We have

$$Lf'(0) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x - 0}{x} = 1,$$

$$Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{0 - 0}{x} = 0.$$

Since $\quad Lf'(0) \neq Rf'(0)$, $f$ is not derivable at $x = 0$.

Obviously $f'(x) = 1$, if $x < 0$ and $f'(x) = 0$, if $x > 0$.

Hence $f$ is derivable at all points except $x = 0$.

**Example 7.** *Let f be defined on* **R** *by setting* $f(x) = 0$, *if* $x \le 0$ *and* $f(x) = x$ *if* $x > 0$. *Show that f is not derivable at* $x = 0$ *and is derivable at every other point.*

**Hint.** Similar to Example 6.

(**Ans.** $Lf'(0) = 0$, $Rf'(0) = 1$. $f'(x) = 0 \ \forall \ x < 0$ and $f'(x) = 1 \ \forall \ x > 0$.)

**Example 8.** *Discuss the derivability of the function :*

$$f(x)=\begin{cases}2x-3, 0\le x\le 2\\ x^2-3, 2<x\le 4\end{cases}$$

*at* $x=2, 4$. [D.U., Maths (H) 1997]

**Solution.** We have

$$Lf'(2)=\lim_{x\to 2-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{(2x-3)-1}{x-2}=2,$$

$$Rf'(2)=\lim_{x\to 2+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{(x^2-3)-1}{x-2}=\lim_{x\to 2}(x+2)=4.$$

Since $Lf'(2)\ne Rf'(2)$, $f$ is not derivable at $x=2$.

$$Lf'(4)=\lim_{x\to 4-}\frac{f(x)-f(4)}{x-4}=\lim_{x\to 4}\frac{(x^2-3)-13}{x-4}=\lim_{x\to 4}(x+4)=8.$$

Since the given function is defined on $[0, 4]$, therefore $f'(4)=Lf'(4)=8$. Hence $f$ is derivable at $x=4$.

**Example 9.** *Discuss the derivability of the function*

$$f(x)=x \text{ for } x<1, f(x)=2-x \text{ for } 1\le x\le 2,$$
$$f(x)=-2+3x-x^2 \text{ for } x>2$$

*at* $x=1, 2$.

**Solution.** $$Lf'(1)=\lim_{x\to 1-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{x-1}{x-1}=1,$$

$$Rf'(1)=\lim_{x\to 1+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{(2-x)-1}{x-1}=-1,$$

$\therefore$ $Lf'(1)\ne Rf'(1)$.

Hence $f$ is not derivable at $x=1$.

Now $$Lf'(2)=\lim_{x\to 2-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{(2-x)-0}{x-2}=-1,$$

$$Rf'(2)=\lim_{x\to 2+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{(-2+3x-x^2)-0}{x-2}$$

$$=\lim_{x\to 2}\frac{-(x-1)(x-2)}{(x-2)}=\lim_{x\to 2}\{-(x-1)\}=-1.$$

$\therefore$ $Lf'(2)=Rf'(2)=-1$.

Hence $f$ is derivable at $x=2$ and $f'(2)=-1$.

**Example 10.** *Prove that the function defined as*

$$f(x)=x\,.\,\frac{e^{1/x}-1}{e^{1/x}+1}, \quad if\ x\ne 0$$
$$=0, \quad if\ x=0$$

*s not derivable at* $x=0$. *Further show that f is continuous at* $x=0$.

[D.U., Maths (H) 1987]

**Solution.** We have

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x} \qquad (\because f(0)=0)$$

$$=\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1}, \text{ which does not exist.}$$

(See Example 11, Chapter 5)

Thus $f$ is not derivable at $x=0$.

Now we show that $f$ is continuous as $x=0$.

$$\lim_{x\to 0-}f(x)=\lim_{x\to 0-}\frac{x(e^{1/x}-1)}{(e^{1/x}+1)}=\frac{0\,.\,(0-1)}{0+1}=0,$$

$$\left(\because \lim_{x\to 0-}e^{1/x}=0\right)$$

$$\lim_{x\to 0+}f(x)=\lim_{x\to 0+}\frac{x(1-e^{-1/x})}{(1+e^{-1/x})}=\frac{0\,.\,(1-0)}{1+0}=0.$$

$$\left(\because \lim_{x\to 0+}e^{-1/x}=0\right)$$

Also $f(0)=0$. Hence $f$ is continuous at $x=0$.

**Example 11.** *Examine the function f where*

$$f(x)=x\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, x\ne 0, f(0)=0$$

*as regards continuity and derivability at the origin.*

[D.U., Maths (H) 1999, 93]

**Solution.** We have

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}},$$

which does not exist. (See Example 19, Chapter 5).

Thus $f$ is not derivable at $x=0$.

Now $$\lim_{x\to 0-}f(x)=\lim_{x\to 0-}\frac{x(e^{1/x}-e^{-1/x})}{(e^{1/x}+e^{-1/x})}=\lim_{x\to 0-}\frac{x(e^{1/x}\,.\,e^{1/x}-1)}{(e^{1/x}\,.\,e^{1/x}+1)}$$

$$=\frac{0\,.\,(0-1)}{0+1}=0,$$

$$\lim_{x\to 0+}f(x)=\lim_{x\to 0+}\frac{x(1-e^{-1/x}\,.\,e^{-1/x})}{(1+e^{-1/x}\,.\,e^{-1/x})}=\frac{0\,.\,(1-0)}{1+0}=0.$$

Also $f(0)=0$. Hence $f$ is continuous at $x=0$.

**Example 12.** *If* $f(x)=x^2\sin(1/x)$ *when* $x\ne 0$ *and* $f(0)=0$, *show that f is derivable for every value of x but the derivative is not continuous for* $x=0$. [D.U., Maths (H) 1991]

Now $$f'(x) = 3x^2 \sin\frac{1}{x} + x^3 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right), \text{ if } x \neq 0$$

$$= 3x\left(x \sin\frac{1}{x}\right) - x\cos\frac{1}{x}$$

$$\therefore \lim_{x\to 0} f'(x) = \lim_{x\to 0}(3x) \lim_{x\to 0}\left(x \sin\frac{1}{x}\right) - \lim_{x\to 0} x\cos\frac{1}{x}$$

$$= 0 \times 0 - 0 = 0 = f'(0), \text{ by } (1)$$

So $\lim\limits_{x\to 0} f'(x) = f'(0)$ implies that $f'(x)$ is continuous at $x = 0$.

**Example 15.** *Let f be defined on* **R** *by setting*

$$f(x) = x^4 \sin(1/x), \text{ if } x \neq 0 \text{ and } f(0) = 0.$$

*Show that $f''(0)$ exists but $f''$ is not continuous at $x = 0$.*

**Solution.** We have

$$f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x^4 \sin(1/x)}{x} = \lim_{x\to 0} x^3 \sin(1/x)$$

$$= \lim_{x\to 0}[x^2 \,.\, x\sin(1/x)] = 0 \times 0 = 0 \qquad [\because \lim_{x\to 0} x\sin(1/x) = 0]$$

Thus $f'(0) = 0$. Now

$$f'(x) = 4x^3 \sin(1/x) - x^2 \cos(1/x), \quad x \neq 0.$$

$$\therefore \quad f''(x) = [12x^2 \sin(1/x) - 4x\cos(1/x)] - [2x\cos(1/x) + \sin(1/x)]$$

**Solution.** (*i*) We have $f(0) = 0$.

$$\text{Now} \quad f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^m \sin \frac{1}{x}}{x} = \lim_{x \to 0} x^{m-1} \sin \frac{1}{x}$$

$$= \lim_{x \to 0} x^{m-2} \cdot \lim_{x \to 0} x \sin \frac{1}{x} \qquad \ldots(1)$$

$$= 0 \times 0 = 0, \text{ if } m - 2 \geq 0 \text{ i.e., } m \geq 2.$$

Hence $f$ is derivable at $x = 0$, if $m \geq 2$.

(*ii*) We have $f'(x) = mx^{m-1} \sin \frac{1}{x} - x^{m-2} \cos \frac{1}{x}$, if $x \neq 0$

and $f'(0) = 0$, if $m \geq 2$, by part (*i*).

$$\text{Now} \quad \lim_{x \to 0} f'(x) = \lim_{x \to 0} \left( mx^{m-1} \sin \frac{1}{x} - x^{m-2} \cos \frac{1}{x} \right)$$

$$= \lim_{x \to 0} x^{m-3} \left[ mx \left( x \sin \frac{1}{x} \right) - x \cos \frac{1}{x} \right]$$

$$= 0, \quad \text{if } m \geq 3.$$

$$\therefore \quad \lim_{x \to 0} f'(x) = f'(0), \text{ if } m \geq 3.$$

Hence $f'$ is continuous at $x = 0$, if $m \geq 3$.

**Example 17.** *Show that the function $f$ defined by*

$$f(x) = \begin{cases} x\,[1 + \frac{1}{3} \sin (\log x^2)], & x \neq 0 \\ 0, & x = 0 \end{cases}$$

*is continuous at $x = 0$ but not derivable at $x = 0$.* [D.U., Maths (H) 1996]

**Solution.** We have

$$|f(x) - f(0)| = |f(x)| = |x|\,|1 + \tfrac{1}{3} \sin (\log x^2)|$$

$$\leq |x| \,.\, [1 + \tfrac{1}{3} |\sin (\log x^2)|]$$

$$\leq |x| \,.\, (1 + \tfrac{1}{3}) = \frac{4}{3} |x - 0|, \text{ since } |\sin \theta| \leq 1 \ \forall\ \theta.$$

Let $\varepsilon > 0$ be given and $\delta = 3\varepsilon/4$. Then

$|f(x) - f(0)| < \varepsilon$, if $|x - 0| < \delta$. Hence $f$ is continuous at $x = 0$.

$$\text{Now} \quad Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{f(x)}{x}$$

$$= \lim_{x \to 0+} \left[ 1 + \frac{1}{3} \sin (\log x^2) \right]$$

$$= 1 + \frac{1}{3} \lim_{x \to 0+} \sin (\log x^2), \text{ which does not exist.}$$

Hence $f$ is not derivable at $x = 0$.

**Example 18.** *Show that the function $f$ defined as*

$$f(x) = |x - 1| + |x + 1| \quad \forall \ x \in \mathbf{R}$$

*is not derivable at the points $x = -1$ and $x = 1$, and is derivable at every other point.*

**Solution.** We have $|x - 1| = x - 1$, if $x \geq 1$

and $|x - 1| = -(x - 1)$, if $x < 1$.

Also $|x + 1| = x + 1$, if $x \geq -1$ and $|x + 1| = -(x + 1)$, if $x < 1$.

$$\therefore \quad f(x) = \begin{cases} -(x - 1) - (x + 1) = -2x, & \text{if } x < -1, & \ldots(1) \\ (x + 1) - (x - 1) = 2, & \text{if } -1 \leq x < 1, & \ldots(2) \\ (x - 1) + x + 1 = 2x, & \text{if } x \geq 1. & \ldots(3) \end{cases}$$

$$\text{Now} \quad Lf'(-1) = \lim_{\substack{x \to -1 \\ (x < -1)}} \frac{f(x) - f(-1)}{x - (-1)}$$

$$= \lim_{x \to -1} \frac{-2x - 2}{x + 1} = \lim_{x \to -1} \frac{-2(x + 1)}{x + 1} = -2, \text{ using (1)}$$

$$Rf'(-1) = \lim_{\substack{x \to -1 \\ (x > -1)}} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \frac{2 - 2}{x + 1} = 0, \text{ using (2).}$$

Since $Lf'(-1) \neq Rf'(-1)$, $f$ is not derivable at $x = -1$.

$$\text{Now} \quad Lf'(1) = \lim_{\substack{x \to 1 \\ (x < 1)}} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{2 - 2}{x - 1} = 0, \text{ using (2)}$$

$$Rf'(1) = \lim_{\substack{x \to 1 \\ (x > 1)}} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{2x - 2}{x - 1} = 2, \text{ using (3)}$$

Since $Lf'(1) \neq Rf'(1)$, $f$ is not derivable at $x = 1$.

From (1), $f'(x) = -2 \ \forall\ x < -1$.

From (2), $f'(x) = 0$, if $-1 < x < 1$.

From (3), $f'(x) = 2 \ \forall\ x > 1$.

## EXERCISES

1. Let $f$ be the function on **R** defined by

$$f(x) = |x - 2| + |x + 2| \text{ for all } x \in \mathbf{R}.$$

Show that $f$ is not derivable at the points $x = -2$ and $x = 2$, and is derivable at every other point. [D.U., Maths (H) 1998]

[**Hint.** Similar to Example 18]

2. Show that $f$ defined by

$$f(x) = |x| + |x - 1| + |x - 2|$$

is continuous but not derivable for 0, 1, 2.

[**Hint.** $f(x)=\begin{cases}-x-(x-1)-(x-2)=-3x+3, & \text{if } x<0\\ x-(x-1)-(x-2)=\ -x+3, & \text{if } 0\le x<1\\ x+(x-1)-(x-2)=\quad x+1, & \text{if } 1\le x<2\\ x+(x-1)+(x-2)=\ 3x-3, & \text{if } x\ge 2.\end{cases}$

3. Let $f$ be defined on **R** by setting

$$f(x)=|x-2|+|x|+|x+2|\ \forall\ x\in \mathbf{R}.$$

Show that $f$ is not derivable at $x=-2$, $x=0$ and $x=2$, and is derivable at every other point. [D.U., B.Sc. (G) 1991]

[**Hint.** $f(x)=\begin{cases}-(x-2)-x-(x+2)=\ -3x, & \text{if } \quad x<-2\\ -(x-2)-x+(x+2)=\ -x+4, & \text{if } -2\le x<0\\ -(x-2)+x+(x+2)=\quad x+4, & \text{if } \quad 0\le x<2\\ \ (x-2)+x+(x+2)=\quad 3x, & \text{if } \quad x\ge 2.\end{cases}$

Then $f'(x)=-3$, if $x<-2$, $f'(x)=-1$, if $-2<x<0$

$f'(x)=1$, if $0<x<2$, $f'(x)=3$, if $x>2$ ].

4. Show that $f(x)=|x+1|+|x-1|$ is continuous but not derivable at $x=-1, 1$. Draw the graph of $f(x)$.

5. Show that the function $f$ defined by

$$f(x)=|x-1|+|x-2|$$

is continuous but not derivable for $x=1, 2$ [D.U., Physics (H) 1993]

6. Show that the function $f$ defined on **R** as follows :

$$f(x)=\begin{cases}x^2, & \text{if } x \text{ is rational,}\\ 0, & \text{if } x \text{ is irrational.}\end{cases}$$

is differentiable at 0 and $f'(0)=0$. Further show that $f$ is not differentiable for all $x\ne 0$.

[**Hint.** We have $f(0)=0$ and

$$\frac{f(x)-f(0)}{x-0}=\frac{f(x)}{x}=\begin{cases}x, & \text{if } x \text{ is rational,}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$$

$$\therefore\quad \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=0. \text{ Hence } f'(0)=0.$$

Since $f$ is not continuous for all $x\ne 0$ [See Example 46, Chapter 5], $f$ is not differentiable for all $x\ne 0$ (Theorem 1) .]

7. Show that the function $f$ defined on $R$ as follows :

$$f(x)=\begin{cases}x, & \text{if } x \text{ is rational,}\\ 0, & \text{if } x \text{ is irrational.}\end{cases}$$

is not differentiable for all $x\in\mathbf{R}$.

[**Hint.** Similar to Ex. 6.]

8. **Given an example of a function which is discontinuous for all $x\ne 0$, but differentiable at $x=0$.**

**[Hint. See Ex. 6 above.]**

9. Show that the function $f$ defined by

$$f(x)=\begin{cases}\dfrac{x}{1+e^{1/x}}, & x\ne 0\\ 0, & x=0\end{cases}$$

is not differentiable at $x=0$.

10. Let $\quad f(x)=\begin{cases}x^2\sin\dfrac{1}{x}, & x\ne 0\\ 0, & x=0\end{cases}$ and $g(x)=x\ \forall\ x\in\mathbf{R}$.

Show that $\lim\limits_{x\to 0}\dfrac{f'(x)}{g'(x)}$ does not exist, but $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}=\dfrac{f'(0)}{g'(0)}$

[**Hint.** See Example 12. $f'(0)=0$ and $g'(0)=1$.

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}x\sin\frac{1}{x}=0 \text{ and } \lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}f'(x)$$

$$=\lim_{x\to 0}[2x\sin(1/x)-\cos(1/x)]=-\lim_{x\to 0}\cos(1/x),$$

which does not exist.]

## 6.2 Algebra of Derivatives

**Theorem 2.** *If $f$ and $g$ be two functions which are defined on $[a, b]$ and derivable at any point $c\in[a, b]$, then their sum $f+g$ is also derivable at $x=c$ and*

$$(f+g)'(c)=f'(c)+g'(c).$$

*Further for each real number $k$, the function $kf$ is also derivable at $c$ and*

$$(kf)'(c)=kf'(c).$$

**Proof.** (*i*) Since $f$ and $g$ are derivable at $c$, therefore

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c) \text{ and } \lim_{x\to c}\frac{g(x)-g(c)}{x-c}=g'(c). \qquad ...(1)$$

Consider $\quad \lim\limits_{x\to c}\dfrac{(f+g)(x)-(f+g)(c)}{x-c}$

$$=\lim_{x\to c}\frac{[f(x)+g(x)]-[f(c)+g(c)]}{x-c}$$

$$=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}+\lim_{x\to c}\frac{g(x)-g(c)}{x-c}$$

$=f'(c)+g'(c)$, using (1).

Hence $f+g$ is derivable at $c$ and $(f+g)'(c)=f'(c)+g'(c)$.

(*ii*) Now $\quad \lim\limits_{x\to c}\dfrac{(kf)(x)-(kf)(c)}{x-c}=\lim\limits_{x\to c}\left[k\left(\dfrac{f(x)-f(c)}{x-c}\right)\right]$

$$=k\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=kf'(c).$$

Hence $kf$ is derivable at $c$ and $(kf)'(c)=kf'(c)$.

**Theorem 3.** *If $f$ and $g$ are two derivable functions at $x = c$, then $fg$ is also derivable at $x = c$ and*

$$(fg)'(c) = f(c)\,g'(c) + g(c)\,f'(c).$$

[D.U., Maths (H) 1999, 92]

**Proof.** Since $f$ and $g$ are derivable at $x = c$, therefore

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ and } g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \qquad \text{...(1)}$$

Since $f$ and $g$ are derivable at $x = c$, so $f$ and $g$ are continuous at $x = c$. Thus

$$\lim_{x \to c} f(x) = f(c), \ \lim_{x \to c} g(x) = g(c). \qquad \text{...(2)}$$

$$\text{Now } \lim_{x \to c} \frac{(fg)(x) - (fg)(c)}{x - c} = \lim_{x \to c} \frac{f(x)\,g(x) - f(c)\,g(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x)\,[g(x) - g(c)] + g(c)\,[f(x) - f(c)]}{x - c}$$

$$= \lim_{x \to c} f(x) \,.\, \lim_{x \to c} \frac{g(x) - g(c)}{x - c} + g(c) \,.\, \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= f(c)\,g'(c) + g(c)\,f'(c), \text{ using (1) and (2).}$$

Hence $fg$ is derivable at $c$ and

$$(fg)'(c) = f(c)\,g'(c) + g(c)\,f'(c).$$

**Theorem 4.** (*i*) *If a function $f$ is derivable at $c$ and if $f(c) \neq 0$, then $1/f$ is also derivable at $c$ and*

$$\left(\frac{1}{f}\right)'(c) = \frac{-f'(c)}{[f(c)]^2}.$$

[D.U., Maths (H) 1994]

(*ii*) *If $f$ and $g$ be two functions on $[a, b]$ and derivable at any point $c \in [a, b]$ and if $g(c) \neq 0$, then $f/g$ is also derivable at $c$ and*

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)\,f'(c) - f(c)\,g'(c)}{[g(c)]^2}.$$

**Proof.** (*i*) Since $f$ is derivable at $c$, so $f$ is continuous at $c$.

Thus $\lim_{x \to c} f(x) = f(c)$. Also $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$.

$$\text{Now} \quad \frac{\frac{1}{f(x)} - \frac{1}{f(c)}}{x - c} = -\frac{f(x) - f(c)}{x - c} \cdot \frac{1}{f(x)} \cdot \frac{1}{f(c)}$$

$$\therefore \quad \lim_{x \to c} \frac{\frac{1}{f(x)} - \frac{1}{f(c)}}{x - c} = -\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \,.\, \lim_{x \to c} \frac{1}{f(x)} \cdot \frac{1}{f(c)}$$

$$= -f'(c) \,.\, \frac{1}{f(c)} \cdot \frac{1}{f(c)} = -\frac{-f'(c)}{\{f(c)\}^2}.$$

Hence $1/f$ is derivable at $c$ and

$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{[f(c)]^2}. \qquad \text{...(1)}$$

(*ii*) Since $g$ is derivable at $c$ and $g(c) \neq 0$, So $1/g$ is derivable at $c$. By Theorem 3, we see that

$$f \text{ and } \frac{1}{g} \text{ are derivable at } c \ \Rightarrow f \,.\, \frac{1}{g} \text{ is derivable at } c.$$

Hence $f/g$ is derivable at $c$. Again by Theorem 3, we have

$$\left(\frac{f}{g}\right)'(c) = f'(c) \,.\, \frac{1}{g(c)} + f(c) \,.\, \left(\frac{1}{g}\right)'(c)$$

$$= f'(c) \,.\, \frac{1}{g(c)} - f(c)\,\frac{g'(c)}{\{g(c)\}^2}, \text{ using (1)}$$

$$\text{Hence} \quad \left(\frac{f}{g}\right)'(c) = \frac{g(c)\,f'(c) - f(c)\,g'(c)}{\{g(c)\}^2}$$

**Theorem 5. (Chain Rule)**

*Let $f : I \to \mathbf{R}$ and $g : J \to \mathbf{R}$ be two real-valued functions defined on the intervals $I$ and $J$, respectively such that $f(I) \subset J$. Let $f$ be differentiable at $c \in I$ and $g$ be differentiable at $f(c)$. Then $gof$ is differentiable at $c$ and*

$$(gof)'(c) = g'(f(c)) \,.\, f'(c).$$

**The proof is omitted.**

**Theorem 6. (Inverse Function Theorem for Derivatives)**

*Let $f$ be a continuous one-to-one function defined on an interval and let $f$ be derivable at $x_0$ with $f'(x_0) \neq 0$. Then the inverse of the function $f$ is derivable at $f(x_0)$ and its derivative at $f(x_0)$ is $1/f'(x_0)$.*

**The proof is omitted.**

**Ex.** *State the Inverse Function Theorem for Derivatives.*

[D.U., Maths (H) 1994]

## EXAMPLES

**Example 19.** *Let $f$ and $g$ be two functions having the same domain $D$. If $fg$ is derivable at $x_0 \in D$, is it necessary that $f$ and $g$ are both derivable at $x_0$? Give argument in support of your answer.* [D.U., Maths (H) 1993]

**Solution.** The function $f$ and $g$ may not be both derivable at $x_0$.

Let $\quad f(x) = |x|$ and $g(x) = -|x| \ \forall \ x \in \mathbf{R}$.

Then $\quad (fg)(x) = f(x)\,g(x) = -|x|^2 = -x^2. \qquad (\because |x|^2 = x^2)$

Obviously, $fg$ is derivable at $x = 0$, since

$$(fg)'(x) = -2x \quad \text{and} \quad (fg)'(0) = 0.$$

But $f$ and $g$ are both not derivable at 0.

**Example 20.** *Let $f$ and $g$ be two functions having the same domain D. If $f \pm g$ and $f/g$ are derivable at $x_0 \in D$, is it necessary that $f$ and $g$ are both derivable at $x_0$. ?*

**Solution.** (*i*) The functions $f$ and $g$ may not be both derivable at $x_0$.

Let $\quad f(x) = x \sin \frac{1}{x}, x \neq 0$ and $f(0) = 0$.

Let $\quad g(x) = -f(x) \; \forall \; x \in \mathbf{R}$.

Then $(f+g)(x) = f(x) + g(x) = 0 \; \forall \; x \in \mathbf{R}$.

$\therefore \quad f+g$ is derivable at 0 and $(f+g)'(0) = 0$.

But $f$ and $g$ are both not derivable at 0.

(*ii*) Let $\quad f(x) = g(x) = |x| \; \forall \; x \in \mathbf{R}$.

Then $(f-g)(x) = f(x) - g(x) = 0 \; \forall \; x \in \mathbf{R}$.

Obviously, $f-g$ is derivable at $x = 0$ and $(f-g)'(0) = 0$.

But $f$ and $g$ are both not derivable at $x = 0$.

(*iii*) Let $f(x) = x \sin \frac{1}{x}, x \neq 0 ; f(0) = 0$. Let $g(x) = \frac{1}{x}, x \neq 0 ;\ g(0) = 1$.

Then $f$ and $g$ are both not derivable at 0. But

$$\left(\frac{f}{g}\right)(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

is derivable at $x = 0$ [Example 13]

**Example 21.** *Let $f(x) = |x|$ and $g(x) = 3|x|$. Show that $f$ and $g$ are not derivable at the origin. But $\lim\limits_{x \to 0} \frac{f(x)}{g(x)}$ exists and is equal to $\lim\limits_{x \to 0} \frac{f'(x)}{g'(x)}$.*

**Solution.** We know that $f$ and $g$ are not derivable at the origin $(x = 0)$.

Now $\quad \lim\limits_{x \to 0} \frac{f(x)}{g(x)} = \lim\limits_{x \to 0} \frac{|x|}{3|x|} = \frac{1}{3}$.

We have $\quad f(x) = x$, if $x \geq 0$ and $f(x) = -x$, if $x < 0$,

and $\quad g(x) = 3x$, if $x \geq 0$ and $g(x) = -3x$, if $x < 0$.

Clearly, $\quad f'(x) = \begin{cases} 1, \text{ if } x > 0 \\ -1, \text{ if } x < 0 \end{cases}, \quad g'(x) = \begin{cases} 3, \text{ if } x > 0 \\ -3, \text{ if } x < 0. \end{cases}$

Now $\quad \lim\limits_{x \to 0-} \frac{f'(x)}{g'(x)} = \lim\limits_{x \to 0} \frac{-1}{-3} = \frac{1}{3}$,

$\quad \lim\limits_{x \to 0+} \frac{f'(x)}{g'(x)} = \lim\limits_{x \to 0} \frac{1}{3} = \frac{1}{3}$.

**Hence** $\quad \lim\limits_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{1}{3} = \lim\limits_{x \to 0} \frac{f(x)}{g(x)}$.

**Example 22.** *Let $f$ and $g$ be two functions with domain D satisfying $g(x) = x f(x) \; \forall \; x \in D$. Show that if $f$ be continuous at $x = 0 \in D$, then $g$ is derivable at $x = 0$.*

**Solution.** We have

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{x f(x) - 0}{x} = \lim_{x \to 0} f(x) = f(0),$$

since $f$ is continuous at $x = 0$.

Hence $g'(0) = f(0)$, which shows that $g$ is derivable at $x = 0$.

## 6.3 Sign of the Derivative

Suppose a function $f$ is defined in $[a, b]$ and $f'(c)$ exists for any point $c \in [a, b]$.

**Let $f'(c) > 0$.**

Since $\quad \lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$, so for $\varepsilon > 0, \exists$ some $\delta > 0$, such that

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon, \text{ when } |x - c| < \delta.$$

$$\left.\begin{array}{l} \Rightarrow \quad f'(c) - \varepsilon < \dfrac{f(x) - f(c)}{x - c} < f'(c) + \varepsilon, \\ \qquad \text{when } x \in \, ]c - \delta, c + \delta[, x \neq c \end{array}\right\} \qquad ..(1)$$

Let us choose $\varepsilon > 0$ so that $\varepsilon < f'(c) \Rightarrow f'(c) > \varepsilon \Rightarrow \mathbf{f'(c) - \varepsilon > 0.}$ ...(2)

From (1), we consider $\quad f'(c) - \varepsilon < \frac{f(x) - f(c)}{x - c}$

$\Rightarrow \frac{f(x) - f(c)}{x - c} > f'(c) - \varepsilon > 0$, using (2)

$\Rightarrow \frac{f(x) - f(c)}{x - c} > 0$, when $x \in \, ]c - \delta, c + \delta[, x \neq c$.

$\Rightarrow f(x) - f(c) > 0$, when $x \in \, ]c, c + \delta[ \qquad (\because x > c)$

and $\quad f(x) - f(c) < 0$, when $x \in \, ]c - \delta, c[$. $\qquad (\because x < c)$

$$\therefore \quad \begin{cases} f(x) > f(c), \text{ when } x \in \, ]c, c + \delta[, \\ f(x) < f(c), \text{ when } x \in \, ]c - \delta, c[. \end{cases}$$

$$\left.\begin{array}{ll} \text{Hence} & f'(c) > 0 \Rightarrow \exists \text{ some } \delta > 0 \text{ such that} \\ & f(x) > f(c), \; \forall \; x \in \, ]c, c + \delta[ \\ \text{and} & f(x) < f(c), \; \forall \; x \in \, ]c - \delta, c[. \end{array}\right\} \qquad ..(A)$$

This property is often stated as follows :

*If $f'(c) > 0$, then $f$ is increasing at $c$.*

**Let $f'(c) < 0$.**

We define a function $g$ as follows :

$$g(x) = -f(x) \quad \forall \quad x \in [a, b].$$

Then $\quad g'(c) = -f'(c) > 0.$

Now $\quad g'(c) > 0 \quad \Rightarrow \quad \exists$ some $\delta > 0$ such that

$$g(x) > g(c) \quad \Rightarrow \quad f(x) < f(c) \ \forall\, x \in ]c, c+\delta[,$$

$$g(x) < g(c) \quad \Rightarrow \quad f(x) > f(c) \ \forall\, x \in ]c-\delta, c[.$$

$$\left.\begin{array}{l}\text{Hence } f'(c) < 0 \Rightarrow \exists \text{ some } \delta > 0 \text{ such that} \\ \quad f(x) < f(c), \ \forall\, x \in ]c, c+\delta[, \\ \quad f(x) > f(c), \ \forall\, x \in ]c-\delta, c[.\end{array}\right\} \qquad ..(B)$$

In other words,

*If $f'(c) < 0$, then $f$ is decreasing at $c$.*

We now consider the end points. It may be shown that

$$\left.\begin{array}{l} f'(a) > 0 \Rightarrow f(x) > f(a), \ \forall\, x \in ]a, a+\delta[, \\ f'(a) < 0 \Rightarrow f(x) < f(a), \ \forall\, x \in ]a, a+\delta[, \end{array}\right\} \qquad ..(C)$$

$$\left.\begin{array}{l} f'(b) > 0 \Rightarrow f(x) < f(b), \ \forall\, x \in ]b-\delta, b[, \\ f'(b) < 0 \Rightarrow f(x) > f(b), \ \forall\, x \in ]b-\delta, b[. \end{array}\right\} \qquad ..(D)$$

**Note.** The students are strongly advised to remember the results (A), (B), (C) and (D) as these will be frequently used.

## 6.4 Darboux's Theorem

**Theorem 7.** *If a function $f$ defined in $[a, b]$ is such that*

*(i) $f$ is derivable in $[a, b]$,*

*(ii) $f'(a)$ and $f'(b)$ are of opposite signs,*

*then there exists some point $c \in ]a, b[$ such that $f'(c) = 0$.*

[D.U., Maths (H) 1990]

**Proof.** Let us suppose that $f'(a) > 0$ and $f'(b) < 0$.

Since $f'(a) > 0 \Rightarrow \exists$ some $\delta_1 > 0$ such that

$$f(x) > f(a), \quad \forall\, x \in ]a, a+\delta_1[. \qquad ...(1)$$

Since $f'(b) < 0 \Rightarrow \exists$ some $\delta_2 > 0$ such that

$$f(x) > f(b), \ \forall\, x \in ]b-\delta_2, b[. \qquad ...(2)$$

Since $f$ is derivable in $[a, b]$, $f$ is continuous in $[a, b]$ and hence bounded therein and the bounds are attained.

Let $\quad K = \sup f$ and $k = \inf f$. Then $\exists$ points $c, d \in [a, b]$ such that

$$f(c) = K \text{ and } f(d) = k. \qquad ...(3)$$

Now $K = \sup f \Rightarrow f(x) \le K, \ \forall\, x \in [a, b]$. $\qquad ...(4)$

We shall show that $c \neq a$ and $c \neq b$.

Let, if possible, $c = a$. Then $f(c) = f(a)$ or $K = f(a)$, using (3)

$\Rightarrow \quad f(x) > K, \quad \forall \quad x \in ]a, a+\delta_1[$, by (1)

This contradicts (4). Thus $c \neq a$.

Similarly, we can show that $c \neq b$. Thus $c \in ]a, b[$.

Finally, we show that $f'(c) = 0$.

**Case I.** Let $f'(c) > 0$.

$\Rightarrow \exists$ some $\delta_3 > 0$ such that $f(x) > f(c), \ \forall\, x \in ]c, c+\delta_3[$

$\Rightarrow f(x) > K, \ \forall\, x \in ]c, c+\delta_2[$, by (3)

This contradicts (4) and so $f'(c) \ngtr 0$.

**Case II.** Let $f'(c) < 0$.

$\Rightarrow \exists$ some $\delta_4 > 0$ such that $f(x) > f(c), \ \forall\, x \in ]c-\delta_4, c[$

$\Rightarrow f(x) > K, \ \forall\, x \in ]c-\delta_4, c[$, by (3)

This contradicts (4) and so $f'(c) \nless 0$. Hence $f'(c) = 0$.

**Remark.** Darboux's Theorem may be restated as follows :

*Let $f$ be defined and derivable on $[a, b]$. If $f'(a) f'(b) < 0$, then there exists some point $c \in ]a, b[$ such that $f'(c) = 0$*

**Proof.** It is clear that

$$f'(a) f'(b) < 0 \Rightarrow f'(a) \text{ and } f'(b) \text{ are of opposite signs.}$$

Now proceed like Theorem 7.

**Theorem 8.** *If $f$ is derivable in $[a, b]$ and $f'(a) \neq f'(b)$, then for each number $k$ lying between $f'(a)$ and $f'(b)$, $\exists$ some point $c \in ]a, b[$ such that*

$$f'(c) = k.$$

[D.U., Maths (H) 1991]

**Proof.** We are given that $\quad f'(a) < k < f'(b)$. $\qquad ...(1)$

Let $\qquad g(x) = f(x) - kx \quad \forall \quad x \in [a, b]$. $\qquad ...(2)$

Since $f$ is derivable in $[a, b]$ and $kx$ is also derivable in $[a, b]$, therefore, by (2), $g$ is derivable in $[a, b]$.

Now $\qquad g'(x) = f'(x) - k \qquad ...(3)$

$\Rightarrow \qquad g'(a) = f'(a) - k < 0$ and $g'(b) = f'(b) - k > 0$, by (1).

Thus $g$ is derivable in $[a, b]$ and $g'(a)$, $g'(b)$ are of opposite signs. So by Darboux's Theorem, $\exists$ some point $c \in ]a, b[$ such that

$$g'(c) = 0$$

$\Rightarrow \quad f'(c) - k = 0$, by (3)

Hence $\quad f'(c) = k$.

### EXAMPLES

**Example 23.** *If $f$ be derivable at a point $c$, then show that $|f|$ is also derivable at $c$, provided $f(c) \neq 0$. Show by means of an example that if $f(c) = 0$, then $f$ may be derivable at $c$ and $|f|$ may not be derivable at $c$.*

**Example 26.** *Let* $f : [-1, 1] \to \mathbf{R}$ *be defined by*

$$f(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 < x \le 1 \end{cases}$$

*Does there exist a function g such that* $g'(x) = f(x)\ \forall\ x \in [-1, 1]$ ?

**Solution.** Let, if possible, there exist a function $g : [-1, 1] \to \mathbf{R}$ such that

$$g'(x) = f(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 < x \le 1. \end{cases}$$

It follows that $g$ is derivable on $[-1, 1]$ and $g'(-1) \neq g'(1)$

$[\because\ g'(-1) = 0$ and $g'(1) = 1]$.

By Theorem 8, $g'$ must assume every value between 0 and 1.

$[\because\ g'(-1) = 0$ and $g'(1) = 1]$.

But this is impossible, since

$$g'(x) = 1 \quad \forall\ x \in\ ]0, 1[.$$

Hence there does not exist any function $g$ such that

$$g'(x) = f(x)\ \forall\ x \in [-1, 1].$$

Now $Lf'(c) = \lim\limits_{x \to c-} \dfrac{f(x) - f(c)}{x - c} \geq 0$, using (5).

(Notice that $x \to c- \Rightarrow x < c$ and so $x - c < 0$)

$\therefore \quad f'(c) \geq 0$, using (4).

Now $Rf'(c) = \lim\limits_{x \to c+} \dfrac{f(x) - f(c)}{x - c} \leq 0$, using (5).

(Notice that $x \to c+ \Rightarrow x > c$ and so $x - c > 0$)

$\therefore \quad f'(c) \leq 0$, using (4).

Hence $f'(c) \geq 0$ and $f'(c) \leq 0 \Rightarrow f'(c) = 0, c \in \,]a, b[$.

This proves the theorem.

**Corollary.** *Let $f$ be a function defined on $[a, a + h]$ such that*

(*i*) *$f$ is continuous on $[a, a + h]$,*

(*ii*) *$f$ is derivable on $]a, a + h[$,*

(*iii*) $f(a) = f(a + h)$.

*Then there exists a real number $\theta$, $0 < \theta < 1$, such that*

$$f'(a + \theta h) = 0.$$

**Proof.** Let $b = a + h$ so that $[a, b] = [a + 0h, a + 1h]$.

Then $c \in \,]a, b[\, = \,]a + 0h, a + 1h[$ must be of the form $c = a + \theta h$, for some $\theta$ satisfying $0 < \theta < 1$.

Hence $\quad f'(c) = 0 \Rightarrow f'(a + \theta h) = 0, 0 < \theta < 1$.

**Note.** *It will help to remember that any point $c \in \,]a, a + h[$ is of the form $c = a + \theta h$, $0 < \theta < 1$.*

**Remarks.** 1. *Rolle's theorem does not hold if $f'$ does not exist even at one point of $]a, b[$.*

Let $\quad f(x) = |x|$ in $[-1, 1]$.

Then $f$ is not derivable at $0 \in \,]-1\ 1[$.

Also $\quad f'(x) \neq 0 \ \forall \ x \in \,]-1, 1[\, - \{0\}$.

2. *Rolle's theorem does not hold if $f$ is not continuous in $[a, b]$.*

Let $$f(x) = \begin{cases} 1, & \text{when } x = 0 \\ x, & \text{when } 0 < x \leq 1. \end{cases}$$

Then $f$ is continuous in $]0, 1]$.

We see that $f'(x) = 1 \neq 0 \ \forall \ x \in \,]0, 1]$.

3. *Rolle's theorem does not hold if $f(a) \neq f(b)$.*

Let $f(x) = x$ in $[1, 2]$. Then $f(1) \neq f(2)$.

We see that $f'(x) = 1 \neq 0 \ \forall \ x \in [1, 2]$.

**4. Geometrical Interpretation of Rolle's Theorem**

*If a continuous curve is drawn in $[a, b]$ such that the ordinates at $a$ and $b$ are equal, then there exists at least one point $c$ lying between $a$ and $b$ such that the tangent at $c$ is parallel to the $x$-axis.*

**Example 4.** *Show that between any two roots of* $e^x \cos x = 1$, *there exists at least one root of* $e^x \sin x - 1 = 0$. [D.U., B.Sc. (G) 1997]

**Solution.** Let $\alpha$ and $\beta$ be any two distinct roots of $e^x \cos x = 1$. ...(1)

$\therefore$ $e^{\alpha} \cos \alpha = 1$ and $e^{\beta} \cos \beta = 1$.

Define a function $f$ as follows :

$$f(x) = e^{-x} - \cos x, \quad \forall \ x \in [\alpha, \beta]. \qquad \text{...(2)}$$

Obviously $f$ is continuous in $[\alpha, \beta]$ and $f$ is derivable in $]\alpha, \beta[$.

Indeed, $f'(x) = -e^{-x} + \sin x \ \forall \ x \in \, ]\alpha, \beta[$.

From (2), $f(\alpha) = e^{-\alpha} - \cos \alpha = \dfrac{1 - e^{\alpha} \cos \alpha}{e^{\alpha}} = 0$, by (1).

Similarly, $f(\beta) = 0$ and so $f(\alpha) = f(\beta)$.

Thus $f$ satisfies all the conditions of Rolle's Theorem in $[\alpha, \beta]$ and so there exists some $\gamma \in \, ]\alpha, \beta[$ such that $f'(\gamma) = 0$.

$$\Rightarrow \quad \sin \gamma - e^{-\gamma} = 0 \Rightarrow e^{\gamma} \sin \gamma - 1 = 0, \alpha < \gamma < \beta.$$

Hence $\gamma$ is a root of $e^x \sin x - 1 = 0, \alpha < \gamma < \beta$.

**Example 5.** *Prove that if* $a_0, a_1, \ldots, a_n$ *are real numbers such that*

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0, \qquad \text{...(1)}$$

*then there exists at least one real number* $x$ *between* 0 *and* 1 *such that*

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0.$$

[D.U., Maths (H) 1992 ; B.Sc. (G) 1992, 91]

**Solution.** Define a function $f$ as follows :

$$f(x) = \frac{a_0}{n+1} x^{n+1} + \frac{a_1}{n} x^n + \ldots + \frac{a_{n-1}}{2} x^2 + a_n x, \quad x \in [0, 1] \qquad \text{...(2)}$$

Since $f$ is a polynomial in $x$, therefore

(*i*) $f$ is continuous in $[0, 1]$,

(*ii*) $f$ is derivable in $]0, 1[$,

(*iii*) $f(0) = f(1)$, since $f(0) = 0$ and $f(1) = 0$, by (1) and (2).

Thus there exists some $x \in \, ]0, 1[$ such that $f'(x) = 0$

or $$\frac{a_0}{(n+1)} \cdot (n+1) x^n + \frac{a_1}{n} . n x^{n-1} + \ldots + a_{n-1} x + \frac{a_{n-1}}{2} . 2x + a_n . 1 = 0$$

Hence $a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0$.

**Example 6.** *Show that there is no real number k for which the equation* $x^3 - 3x + k = 0$ *has two distinct roots in* $[0, 1]$. [D.U., B.Sc. (G) 1999]

**Solution.** Suppose, on the contrary, that there is a real number $k$ for which the given equation has two distinct roots $\alpha$ and $\beta$ in $[0, 1]$, where $\alpha < \beta$. Obviously, $[\alpha, \beta] \subset [0, 1]$. ...(1)

**Example 11.** *If $p(x)$ is polynomial and $k \in \mathbf{R}$, prove that between any two real roots of $p(x) = 0$, there is a roots of $p'(x) + k\,p(x) = 0$.*

**Solution.** Let $\quad f(x) = e^{kx}\,p(x), x \in \mathbf{R}$. ...(1)

Let $\alpha$ and $\beta$ be any two real roots of $p(x)$, where $\alpha < \beta$. Then $p(\alpha) = 0 = p(\beta) \Rightarrow f(\alpha) = 0 = f(\beta)$, by (1).

Obviously, (*i*) $f$ is continuous in $[\alpha, \beta]$, (*ii*) $f$ is derivable in $]\alpha, \beta[$ and (*iii*) $f(\alpha) = f(\beta)$. Hence, by Rolle's theorem, there exists some $\gamma \in\, ]\alpha, \beta[$ such that $f'(\gamma) = 0$

From (1), $\quad f'(x) = k\,e^{kx}\,p(x) + e^{kx}\,p'(x) = e^{kx}\,[p'(x) + kp(x)]$.

$$\therefore \qquad 0 = f'(\gamma) = e^{k\gamma}[p'(\gamma) + k\,p(\gamma)]$$

$$\Rightarrow p'(\gamma) + k\,p(\gamma) = 0, \text{ as } e^{k\gamma} \neq 0.$$

Hence $\gamma$ is a root of $p'(x) + kp(x) = 0$, where $\alpha < \gamma < \beta$.

**Example 12.** *Prove that if $P$ be any polynomial and $P'$ the derivative of $P$, then between any two consecutive zeros of $P'$, there lies at the most one zero of $P$.* [D.U., Maths (H) 1998]

**Solution.** Let $\alpha$ and $\beta$ be any two consecutive zeros of $P'$. Then

$$P'(\alpha) = 0 = P'(\beta) \text{ and } P'(x) \neq 0 \ \forall \ x \in\, ]\alpha, \beta[. \qquad ...(1)$$

Let, if possible, there exist two zeros $c$ and $d$ of $P$, where $\alpha < c < d < \beta$. Then $P(c) = 0 = P(d)$. Since $P$ is a polynomial, $P$ satisfies all the conditions of Rolle's theorem in $[c, d]$. Consequently, there exists some point $x_0 \in\, ]c, d[$ (*i.e.*, $x_0 \in\, ]\alpha, \beta[$ ) such that $P'(x_0) = 0$, which contradicts (1). Hence our assumption is wrong and so there exists at the most one zero of $P$, which lies between two consecutive zeros of $P'$.

## EXERCISES

1. Verify Rolle's Theorem for $f(x) = x^3 - 4x$ or $[-2, 2]$.
   (**Ans.** $c = \pm 2/\sqrt{3}$)
2. Verify Rolle's Theorem for $f(x) = \cos x$ in $[-\pi/2, \pi/2]$. (**Ans.** $c = 0$)
3. Verify Rolle's Theorem for $f(x) = x^3 - 6x^2 + 11x - 6$ in $[1, 3]$.
   (**Ans.** $c = 2 \pm \frac{1}{\sqrt{3}}$.)
4. Examine the validity of the hypotheses and the conclusion of Rolle's theorem for the functions :
   (*i*) $f(x) = |x|, -1 \leq x \leq 1$.
   (*ii*) $f(x) = x^{5/2}, 1 \leq x \leq 5$. (**Ans.** (*i*) and (*ii*). Both not valid)
5. Verify Rolle's theorem for $f(x) = e^x \sin x$ in $[0, \pi]$. (**Ans.** $c = 3\pi/4$)
6. Discuss the applicability of Rolle's theorem to the function

$$f(x) = \log\left[\frac{x^2 + ab}{(a+b)\,x}\right] \text{ in } [a, b].$$

**7.** Prove that if $p(x)$ is a polynomial, then between any two roots of $p(x) = 0$ lies a root of $p'(x) = 0$.

[**Hint.** Apply Rolle's theorem to $p(x)$ in $[\alpha, \beta]$, $p(\alpha) = 0 = p(\beta)$.]

**8.** Prove that there is no $k \in \mathbf{R}$ for which the equation $x^3 - 12x + k = 0$ has two real roots in $[-1, 1]$.

[**Hint.** Similar to Example 6.]

**9.** If $a_0 + a_1 + \dots + a_n = 0$, where $a_0, a_1, \dots, a_n \in \mathbf{R}$ ; show that the equation $a_0 + 2a_1 x + \dots + (n+1) a_n x^n = 0$ has at least one real root in $(0, 1)$.

[**Hint.** Let $f(x) = a_0 x + a_1 x^2 + a_2 x^3 + \dots + a_n x^{n+1}$ in $[0, 1]$. Now proceed like Example 5.]

**10.** Show that $x = 0$ is the only real root of the equation $e^x = 1 + x$.

[**Hint.** Let $f(x) = e^x - x - 1 \ \forall\ x \in [0, \alpha]$, where $\alpha \in \mathbf{R}$ is a root of $f(x) = 0$. Obviously, $f(0) = 0$. By Rolle's theorem, there exists some $c \in ]0, \alpha[$ such that $f'(c) = 0 \Rightarrow e^c - 1 = 0 \Rightarrow c = 0$, a contradiction.]

**11.** Show that between two real roots of $e^x \sin x + 1 = 0$, there exists a real root of $e^x \cos x = 1$.

[**Hint.** Similar to Example 4. Let $f(x) = \sin x + e^{-x}$.]

**12.** If $f(x)$, $g(x)$ are differentiable on $(a, b)$ and are continuous on $[a, b]$ and $f(a) = f(b) = 0$, then there exists a point $c \in (a, b)$ such that

$$f'(c) + f(c)\, g'(c) = 0.$$

[**Hint.** Let $\phi(x) = f(x)\, e^{g(x)}$. Then $\phi$ satisfies the conditions of Rolle's theorem in $[a, b]$ and so $\phi'(c) = 0$ for some $c \in (a, b)$ *i.e.*, $f'(c)\, e^{g(c)} + f(c)\, e^{g(c)} \cdot g'(c) = 0$. Hence $f'(c) + f(c)\, g'(c) = 0$.]

**13.** Prove that for any real number $k$, the polynomial given by $f(x) = x^3 + x + k$ has exactly one real root.

[**Hint.** Let, if possible, $f(\alpha) = 0 = f(\beta)$ ; $\alpha < \beta$. By Rolle's Theorem, $f'(\gamma) = 0$ for some $\gamma \in ]\alpha, \beta[ \Rightarrow 3\gamma^2 + 1 = 0 \Rightarrow \gamma^2 = -\frac{1}{3} < 0$, which is impossible.]

**14.** Using $f(x) = (x - 4) \log x$, prove that the equation $x \log x = 4 - x$ is satisfied by at least one value of $x$ in $[1, 4]$.

[**Hint.** Apply Rolle's theorem to $f(x)$ in $[1, 4]$. There exists some $c \in ]1, 4[$ such that $f'(c) = 0$. We have

$$f'(x) = \frac{x-4}{x} + \log x \Rightarrow c \log c = 4 - c,\ 1 < c < 4.]$$

## 7.2 Lagrange's Mean Value Theorem

***Let f be a function defined on [a, b] such that***

*(i) f is continuous in [a, b],*

*(ii) f is derivable in ]a, b[.*

***Then there exists some $c \in ]a, b[$ such that***

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

[D.U., B.Sc. (G) 1994, 92, 90 ; B.A. (P) 1992, 91]

**Proof.** Define a function $\phi$ on $[a, b]$ as follows :

$$\phi(x) = f(x) + Ax,$$

where $A$ is a constant to be chosen such that $\phi(a) = \phi(b)$

*i.e.,* $$f(a) + Aa = f(b) + Ab$$

*i.e.,* $$A = -\frac{f(b) - f(a)}{b - a}. \quad \text{...(1)}$$

Now $\phi$ satisfies the following conditions :

(*i*) $\phi$ is continuous in $[a, b]$, since $f(x)$ and $Ax$ are both continuous in $[a, b]$ imply $f(x) + Ax$ is continuous in $[a, b]$

(*ii*) $\phi$ is derivable in $]a, b[$, since $f(x)$ and $Ax$ are both derivable in $]a, b[$ imply $f(x) + Ax$ is derivable in $]a, b[$

(*iii*) $\phi(a) = \phi(b)$.

Since all the conditions of Rolle's Theorem are satisfied, there exists some $c \in ]a, b[$ such that $\phi'(c) = 0$.

**Now** $\quad \phi'(x) = f'(x) + A \Rightarrow f'(c) + A = 0 \Rightarrow f'(c) = -A$.

**Hence** $\quad f'(c) = \dfrac{f(b) - f(a}{b - a}$, using (1).

**Corollary 1.** *If f satisfies the conditions of Langrange's mean value theorem and if $f'(x) = 0\ \forall\ x \in ]a, b[$, then f is constant on [a, b].*

**Proof.** Let $x_1$ and $x_2$ be any two points of $[a, b]$ such that $x_1 < x_2$.

By Lagrange's mean value theorem,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0, \quad \text{where } c \in ]x_1, x_2[. \quad \text{...(3)}$$

Thus $f(x_1) = f(x_2)\ \forall\ x_1, x_2 \in [a, b]$ and so $f$ is constant on $[a, b]$.

**Corollary 2.** *If f satisfies the conditions of Lagrange's mean value theorem and if $f'(x) > 0$ ($f'(x) < 0$) $\forall\ x \in ]a, b[$, then f is strictly increasing (strictly decreasing) in [a, b].*

**Proof.** Let $x_1$ and $x_2$ be any two points of $[a, b]$ such that $x_1 < x_2$.

Let $\quad f'(x) > 0\ \forall\ x \in ]a, b[$.

By Lagrange's mean value theorem,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \text{ for some } c \in ]x_1, x_2[$$

$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c) > 0$, since $f'(c) > 0$ and $x_2 - x_1 > 0$

$\Rightarrow f(x_2) > f(x_1)$, when $x_2 > x_1$ ; $x_1, x_2 \in [a, b]$.

**Hence** $f$ is strictly increasing in $[a, b]$.

**Similarly,** we can show that if $f'(x) < 0\ \forall\ x \in ]a, b[$, then $f$ is strictly **decreasing in $[a, b]$.**

**Corollary 3.** *If two functions f and g are*

(*i*) *continuous in* $[a, b]$,

(*ii*) *derivable in* $]a, b[$,

(*iii*) $f'(x) = g'(x)$, $\forall\, x \in\, ]a, b[$.

*Then f and g differ by a constant.*

**Proof.** Define a function $\phi$ as follows :

$$\phi(x) = f(x) - g(x),\ \forall\, x \in [a, b].$$

The function $\phi$, being the difference of two continuous functions in $[a, b]$ and differentiable in $]a, b[$, is also continuous in $[a, b]$ and differentiable in $]a, b[$, respectively.

Now $\phi'(x) = f'(x) - g'(x) = 0$, $\forall\, x \in\, ]a, b[$, by (*iii*).

Thus $\phi = f - g$ satisfies all the conditions of Corollary 1 and so $\phi$ is constant in $[a, b]$. Hence $f$ and $g$ differ by a constant.

**Corollary 4. Another statement of Lagrange's Mean Value Theorem**

*If a function f defined in* $[a, a+h]$ *is such that*

(*i*) *f is continuous in* $[a, a+h]$,

(*ii*) *f is derivable in* $]a, a+h[$.

*Then there exists some* $\theta \in\, ]0, 1[$ *such that*

$$f(a+h) - f(a) = hf'(a+\theta h).$$

**Proof.** Take $b = a + h$ and $c \in\, ]a, a+h[$ as $c = a + \theta h$, $0 < \theta < 1$.

**Corollary 5. Geometrical Interpretation of Lagrange's Mean Value Theorem**

If $A\,(a, f(a))$ and $B\,(b, f(b))$ are any two points on a continuous curve $y = f(x)$, then the slope of the chord $AB$ is

$$\frac{f(b) - f(a)}{b - a}.$$

Hence geometrically speaking, Lagrange's mean value theorem says : *There exists a point* $x = c$ *on the curve* $y = f(x)$ *lying between A and B at which the tangent to the curve is parallel to the chord AB.*

## EXAMPLES

**Example 13.** *Verify Lagrange's Mean Value Theorem for the function* $f(x) = \sqrt{x^2 - 4}$ *in* $[2, 4]$. [D.U. B.Sc. (G) 1994]

**Solution.** The function $f(x) = \sqrt{x^2 - 4}$ is an algebraic function of $x$, so $f$ is continuous in $[2, 4]$. Also $f$ is derivable in $]2, 4[$.

Now $\quad f'(x) = \dfrac{x}{\sqrt{x^2 - 4}} \quad \forall\ x \in\, ]2, 4[$.

Since $f$ satisfies all the conditions of Lagrange's mean value theorem, therefore, there exists some $c \in\, ]2, 4[$ such that

$$\frac{f(4) - f(2)}{4 - 2} = f'(c)$$

$$\Rightarrow 2\sqrt{3} - 0 = 2 . \frac{c}{\sqrt{c^2 - 4}}$$

$$\Rightarrow \sqrt{3} = \frac{c}{\sqrt{c^2 - 4}} \Rightarrow 3(c^2 - 4) = c^2 \Rightarrow 2c^2 = 12 \Rightarrow c = \pm\sqrt{6}.$$

Obviously, $\quad c = \sqrt{6} \in\, ]2, 4[$.

Hence the Lagrange's mean value theorem is verified.

**Example 14.** *Verify Lagrange's Mean Value Theorem for the function* $f(x) = x(x-1)(x-2)$ *in* $[0, \frac{1}{2}]$. [D.U., B.A. (P) 1992]

**Solution.** Since $f$ is a polynomial, $f$ is continuous in $[0, \frac{1}{2}]$ and derivable in $]0, \frac{1}{2}[$. Thus there exists $c \in\, ]0, \frac{1}{2}[$ such that

$$\frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0} = f'(c). \qquad \ldots(1)$$

Now $\quad f'(x) = (x-1)(x-2) + x(x-1) + x(x-2)$

or $\quad f'(x) = 3x^2 - 6x + 2$. Now $f(0) = 0, f(\frac{1}{2}) = \frac{3}{8}$.

From (1), $\quad \frac{3}{8} = \frac{1}{2}(3c^2 - 6c + 2)$

or $\quad 12c^2 - 24c + 5 = 0$ or $c = \dfrac{6 \pm \sqrt{21}}{6}$

Now $\quad c = \dfrac{6 + \sqrt{21}}{6} \notin\, ]0, 1[$, $\ c = \dfrac{6 - \sqrt{21}}{6} \in\, ]0, 1[$.

Thus $\quad c = \dfrac{6 - \sqrt{21}}{6}$ and the theorem is verified.

**Example 15.** *Prove that for any quadratic function* $px^2 + qx + r$, *the value of* $\theta$ *in Lagrange's theorem is always* $\frac{1}{2}$, *whatever p, q, r, a, h may be.*

**Solution.** Let $f(x) = px^2 + qx + r$ in $[a, a+h]$.

Since $f$ is a polynomial, $f$ satisfies the conditions of Lagrange's mean value theorem and so there exists $\theta$, $0 < \theta < 1$, satisfying

$$f(a+h) - f(a) = hf'(a+\theta h) \qquad \text{[See Corollary 4 of Sec 7.2]}$$

*i.e.*, $\quad p(a+h)^2 + q(a+h) + r - pa^2 - qa - r = h\,[\,2p(a+\theta h) + q\,]$

On simplifying this equation, we obtain $\theta = \frac{1}{2}$.

**Example 16.** *Let f be defined and continuous in* $[a-h, a+h]$ *and derivable in* $]a-h, a+h[$. *Prove that there is a real number* $\theta$ *between* 0 *and* 1 *such that*

$$f(a+h) - f(a-h) = h[\,f'(a+\theta h) + f'(a-\theta h)].$$

[D.U., B.Sc. (G) 1995 ; Maths (H) 1992]

**Solution.** Define $\ \phi(x) = f(a+hx) - f(a-hx)$ on $[0, 1]$. $\qquad \ldots(1)$

$\therefore \quad \phi'(x) = hf'(a+hx) + hf'(a-hx). \qquad \ldots(2)$

**Solution.** We shall apply Lagrange's mean value theorem to the function $f(x) = e^x$ in the interval $[0, x]$. The function $f(x) = e^x$ is derivable in $[0, x]$ and, therefore, there exists some $c, 0 < c < x$, such that

$$\frac{e^x - e^0}{x - 0} = e^c \quad \text{or} \quad \frac{1}{x}(e^x - 1) = e^c. \qquad \text{...(1)}$$

Since $0 < c < x$, so $1 = e^0 < e^c < e^x$. ...(2)

From (1) and (2), $1 < \frac{1}{x}(e^x - 1) < e^x$ or $x < e^x - 1 < xe^x$. $(\because x > 0)$

Hence $1 + x < e^x < 1 + xe^x \ \forall \ x > 0$.

**Example 20.** *Show that*

$$\frac{v - u}{1 + v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v - u}{1 + u^2}, \quad \text{if } 0 < u < v,$$

*and deduce that* $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$. [D.U., Maths (H) 1995, 91]

**Solution.** Applying Lagrange's mean value theorem to the function

$f(x) = \tan^{-1} x$ in $[u, v]$, we obtain

$$\frac{f(v) - f(u)}{v - u} = f'(c) \text{ for some } c \in \,]u, v[$$

or
$$\frac{\tan^{-1} v - \tan^{-1} u}{v - u} = \frac{1}{1 + c^2}, \text{ for } u < c < v. \qquad \text{...(1)}$$

Now $c > u \Rightarrow 1 + c^2 > 1 + u^2 \Rightarrow \frac{1}{1 + c^2} < \frac{1}{1 + u^2}$. ...(2)

Again $c < v \Rightarrow 1 + c^2 < 1 + v^2 \Rightarrow \frac{1}{1 + c^2} > \frac{1}{1 + v^2}$. ...(3)

From (1), (2), (3) ; we obtain

$$\frac{1}{1 + v^2} < \frac{\tan^{-1} v - \tan^{-1} u}{v - u} < \frac{1}{1 + u^2}.$$

Hence
$$\frac{v - u}{1 + v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v - u}{1 + u^2}. \qquad \text{...(4)}$$

$(\because u < v \Rightarrow v - u > 0)$

Taking $u = 1$ and $v = \frac{4}{3}$ in (4), we obtain

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{1}{6}.$$

or
$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}.$$

Hence $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$.

**4.** If $a$ and $b$ are distinct real numbers, show that there exists a real number $c$ between $a$ and $b$ such that $a^2 + ab + b^2 = 3c^2$.

**[Hint.** Apply L.M.V. Theorem to $f(x) = x^3$ in $[a, b]$ to get

$$\frac{b^3 - a^3}{b - a} = 3c^2 \Rightarrow b^2 + ba + a^2 = 3c^2, \ c \in \,]a, b[\ ].$$

**5.** Prove that $|\sin x - \sin y| \le |x - y| \ \forall \ x, y \in \mathbf{R}$.

**[Hint.** Take $f(x) = \sin x$ and proceed like Example 21. Use $|\cos\theta| \le 1 \ \forall \ \theta$.]

## 7.3 Cauchy's Mean Value Theorem

*Let $f$ and $g$ be two functions defined on $[a, b]$ such that*

*(i) $f$ and $g$ are continuous in $[a, b]$,*

*(ii) $f$ and $g$ are derivable in $]a, b[$,*

*(iii) $g'(x) \ne 0$ for each $x \in \,]a, b[$.*

*Then there exists at least one point $c \in \,]a, b[$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

[D.U., Maths (H) 1996 ; B.Sc. (G) 1997, 96 ; B.A. (P) 1997, 95, 90]

**Proof.** The given conditions imply that $g(b) \ne g(a)$, for if $g(b) = g(a)$, then $g$ satisfies the conditions of the Rolle's theorem and so there exists some $x_0 \in \,]a, b[$ such that $g'(x_0) = 0$, which contradicts *(iii)*.

Let $\phi(x) = f(x) + A\,g(x) \ \forall \ x \in [a, b]$, ...*(iv)*

where $A$ is a constant to be chosen suitably. We observe that

*(i)* $\phi$ is continuous in $[a, b]$, since $f$ and $g$ are continuous in $[a, b]$.

*(ii)* $\phi$ is derivable in $]a, b[$, since $f$ and $g$ are derivable in $]a, b[$.

Choose the constant $A$ such that $\phi(a) = \phi(b)$

$\Rightarrow \quad f(a) + A \,.\, g(a) = f(b) + A \,.\, g(b)$, using *(iv)*

$$\Rightarrow \qquad A = -\frac{f(b) - f(a)}{g(b) - g(a)}.$$

Since $\phi$ satisfies all the conditions of Rolle's theorem, there exists some $c \in \,]a, b[$ such that $\phi'(c) = 0$

$\Rightarrow \quad f'(c) + A\,g'(c) = 0$, using *(iv)*

$$\Rightarrow \qquad A = -\frac{f'(c)}{g'(c)}.$$

Hence $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

**Remark.** If we take $g(x) = x \ \forall \ x \in \,]a, b[$ in Cauchy's mean value theorem, we obtain

$$\frac{f(b) - f(a)}{b - c} = f'(c) \qquad (\because \ g'(c) = 1),$$

**Solution.** *(i)* Clearly $f(x) = x^2$, $g(x) = x^3$ are continuous in $[1, 2]$ and derivable in $]1, 2[$. Further $g'(x) = 3x^2 \neq 0 \; \forall \; x \in \; ]1, 2[$. Thus the conditions of the Cauchy's mean value theorem are satisfied and so there exists some point $c \in \; ]1, 2[$ such that

$$\frac{f(2) - f(1)}{g(2) - f(1)} = \frac{f'(c)}{g'(c)}, \; 1 < c < 2$$

or $$\frac{4 - 1}{8 - 1} = \frac{2c}{3c^2} \quad \text{or} \quad \frac{3}{7} = \frac{2}{3c}$$

or $$c = \frac{14}{9} = 1.55 \in \; ]1, 2[.$$

Hence Cauchy's mean value theorem is verified.

*(ii)* Clearly $f(x) = \sin x$, $g(x) = \cos x$ are continuous in $[-\pi/2, 0]$ and derivable in $]-\pi/2, 0[$. Further $g'(x) = -\sin x \neq 0$ for all $x \in \; ]-\pi/2, 0[$. Thus the conditions of the Cauchy's mean value theorem are satisfied and so there exists some point $c \in \; ]-\pi/2, 0[$ such that

$$\frac{f(0) - f(-\pi/2)}{g(0) - g(-\pi/2)} = \frac{f'(c)}{g'(c)} \quad \text{or} \quad \frac{0 + 1}{1 - 0} = \frac{\cos c}{-\sin c}$$

or $\tan c = -1$, which gives $c = -\pi/4 \in \; ]-\pi/2, 0[$.

Hence Cauchy's mean value theorem is verified.

*(iii)* Clearly $f(x) = e^x$ and $g(x) = e^{-x}$ satisfy the conditions of the Cauchy's mean value theorem and so there exists some $c \in \; ]0, 1[$ such that

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f'(c)}{g'(c)} \quad \text{or} \quad = \frac{e - 1}{e^{-1} - 1} = \frac{e^c}{-e^{-c}}$$

or $$\frac{e(e - 1)}{1 - e} = -e^{2c} \quad \text{or} \quad e^{2c} = e^1$$

or $$2c = 1 \quad \text{or} \quad c = \frac{1}{2} \in \; ]0, 1[.$$

Hence Cauchy's mean value theorem is verified.

**Example 24.** *Show that*

$$\frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \theta, \quad \textit{where } 0 < \alpha < \theta < \beta < \frac{\pi}{2}.$$

[D.U., Maths (H) 1996 ; B.Sc. (G) 1998]

**Solution.**

Let $f(x) = \sin x$, $g(x) = \cos x \; \forall \; x \in [\alpha, \beta]$, $0 < \alpha < \theta < \beta < \frac{\pi}{2}$.

Then $f(x)$ and $g(x)$ are continuous in $[\alpha, \beta]$ and derivable in $]\alpha, \beta[$. Further $g'(x) = -\sin x \neq 0 \quad \forall \; x \in \; ]\alpha, \beta[ \; \subset \; ]0, \pi/2[$.

By Cauchy's mean value theorem,

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}, \text{ for some } \theta \in \; ]\alpha, \beta[.$$

$$\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}, \quad \alpha < \theta < \beta.$$

*i.e.*,

$$\text{Hence} \quad \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \quad \text{where } 0 < \alpha < \theta < \beta < \frac{\pi}{2}.$$

**Example 25.** *Let the function $f$ be continuous in $[a, b]$ and derivable in $]a, b[$. Show that there exists a number $c$ in $]a, b[$ such that*

$$2c\,[f(a) - f(b)] = f'(c)\,[a^2 - b^2].$$

**Solution.** Let $g(x) = x^2 \;\; \forall \;\; x \in [a, b], a < b$.

Then $f$ and $g$ are continuous in $[a, b]$ and derivable in $]a, b[$. Further $g'(x) = 2x \neq ]a, b[$. By Cauchy's mean value theorem,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \quad \text{for some } c \in \,]a, b[$$

*i.e.*,

$$\frac{f(b) - f(a)}{b^2 - a^2} = \frac{f'(c)}{2c}.$$

Hence $2c\,[f(a) - f(b)] = f'(c)[a^2 - b^2],\; c \in \,]a, b[$ .

**Example 26.** *If $f'$ and $g'$ exist for all $x \in [a, b]$ and if $g'(x) \neq 0 \;\forall\; x \in \,]a, b[$, then prove that for some $c \in \,]a, b[$*

$$\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}.$$

[D.U., B.Sc. (G) 1993 ; Maths (H) 1998]

**Solution.** Define a function $\psi$ on $[a, b]$ as follows :

$$\psi(x) = f(x)\,g(x) - f(a)\,g(x) - g(b)\,f(x), \; \forall\; x \in \,]a, b[.$$

Since $f$ and $g$ are derivable in $[a, b]$, so $\psi$ is continuous in $[a, b]$ and derivable in $]a, b[$.

Also $\quad \psi(a) = \psi(b) = -f(a)\,g(b)$.

Since $\psi$ satisfies all the conditions of Rolle's Theorem, therefore, there exists some $c \in \,]a, b[$ such that $\psi'(c) = 0$. We have

$$\psi'(x) = f'(x)\,g(x) + f(x)\,g'(x) - f(a)\,g'(x) - g(b)\,f'(x).$$

$\therefore \qquad \psi'(c) = 0$ gives us

$$f'(c)\,g(c) + f(c)\,g'(c) - f(a)\,g'(c) - g(b)\,f'(c) = 0$$

or

$$g'(c)\,\{f(c) - f(a)\} = f'(c)\,\{g(b) - g(c)\}.$$

$$\text{Hence} \quad \frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}.$$

**Example 27.** *If $f'$ and $g'$ are continuous and differentiable on $[a, b]$, then show that for $a < c < b$,*

$$\frac{f(b) - f(a) - (b - a)f'(a)}{g(b) - g(a) - (b - a)g'(a)} = \frac{f''(c)}{g''(c)}.$$

[D.U., Maths (H) 1991]

**Solution.** Define a function $\phi$ on $[a, b]$ as follows :

$$\phi(x) = \{f(x) + (b - x)\,f'(x)\} + A\,\{g(x) + (b - x)\,g'(x)\}, \qquad \ldots(1)$$

where $A$ is a constant to be determined by

$$\phi(a) = \phi(b) \qquad \ldots(2)$$

or

$$\{f(a) + (b - a)\,f'(a)\} + A\,\{g(a) + (b - a)\,g'(a)\} = f(b) + Ag(b)$$

or

$$-A = \frac{f(b) - f(a) - (b - a)\,f'(a)}{g(b) - g(a) - (b - a)\,g'(a)}. \qquad \ldots(3)$$

Since $f'$ and $g'$ are continuous and derivable in $[a, b]$, so by (1), $\phi$ is continuous and derivable in $[a, b]$. By (2), $\phi(a) = \phi(b)$. Thus $\phi$ satisfies the conditions of Rolle's theorem and so there exists some point $c \in \,]a, b[$ such that $\phi'(c) = 0$. From (1), we have

$$\phi'(x) = \{f'(x) - f'(x) + (b - x)f''(x)\} + A\,\{g'(x) - g'(x) + (b - x)g''(x)\}$$

$$\therefore \qquad \phi'(c) = 0 \Rightarrow (b - c)\,f''(c) + A\,(b - c)\,g''(c) = 0$$

$$\Rightarrow \qquad -A = \frac{f''(c)}{g''(c)}, \text{ since } a < c < b \Rightarrow b - c \neq 0.$$

Substituting this value of $-A$ in (3), the required result is proved.

**Example 28.** *If a function $f$ is such that its derivative $f'$ is continuous on $[a, b]$ and derivable on $]a, b[$, then show that there exists a number $c$ between $a$ and $b$ such that*

$$f(b) = f(a) + (b - a)\,f'(a) + \tfrac{1}{2}\,(b - a)^2 f''(c).$$

[D.U., Maths (H) 1997]

**Solution.** Define a function $\phi$ on $[a, b]$ as follows :

$$\phi(x) = f(b) - f(x) - (b - x)\,f'(x) - (b - x)^2 A, \qquad \ldots(1)$$

where $A$ is a constant to be determined by

$$\phi(a) = \phi(b) \qquad \ldots(2)$$

*i.e.*, $\quad f(b) - f(a) - (b - a)\,f'(a) - (b - a)^2 A = 0$

*i.e.*, $\quad f(b) = f(a) + (b - a)\,f'(a) + (b - a)^2 A . \qquad \ldots(3)$

Since $f'$ is continuous on $[a, b]$, so $f$ is also continuous on $[a, b]$ Also $(b - x), (b - x)^2$ are continuous on $[a, b]$.

Thus by (1), $\phi$ is continuous on $[a, b]$. Similarly, $\phi$ is derivable on $]a, b[$. Further $\phi(a) = \varphi(b)$, by (2).

Thus $\phi$ satisfies the conditions of Rolle's Theorem and so there exists some point $c \in \,]a, b[$ such that $\phi'(c) = 0. \qquad \ldots(4)$

From (1), $\quad \phi'(x) = -f'(x) - \{-f'(x) + (b - x)\,f''(x)\} + 2(b - x)\,A$

or

$$\phi'(x) = -(b - x)\,f''(x) + 2\,(b - x)\,A. \qquad \ldots(5)$$

$$\therefore \qquad \phi'(c) = 0 \quad \Rightarrow \; -(b - c)\,f''(c) + 2\,(b - c)\,A = 0$$

$$\Rightarrow A = \tfrac{1}{2} f''(c), \text{ since } a < c < b \Rightarrow b - c \neq 0.$$

Substituting this value of $A$ in (3), the result is proved.

**Example 29.** *If a function $f$ is such that $f$ is continuous on $[a, b]$ and derivable in $(a, b)$, show that there exists a real number $\theta$, $0 < \theta < 1$ such that*

$$f(b) = f(a) + (b - a)\,f'(a) + \frac{(b - a)^2}{2\,!}\,f''(a + \theta\,(b - a)).$$

[D.U., B.C.A. 1999]

**Solution.** Refer to Example 28. We take $c = a + \theta(b-a), 0 < \theta < 1$ in equation (4) *i.e.*, $\phi'(a + \theta(b-a)) = 0$.

[Notice that for $\theta = 0, c = a$ and for $\theta = 1, c = b$. Consequently, $c = a + \theta(b-a) \in\ ]a, b[$, if $0 < \theta < 1$.]

From (5), $\phi'(x) = (b-x)\,[2A - f''(x)]$.

or
$$0 = \phi'(a + \theta(b-a)) = (b-a)(1-\theta)\,[2A - f''(a + \theta(b-a))].$$

$$\therefore \quad A = \tfrac{1}{2} f''(a + \theta(b-a)), \text{ since } a \neq b \text{ and } 0 < \theta < 1.$$

Putting this value of $A$ in (3), we get the desired result.

**Example 30.** *If $f'$ is continuous on $[a, a+h]$ and derivable on $]a, a+h[$, then prove that there exists a real number $c$ between $a$ and $a+h$ such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(c). \qquad \text{[D.U., Maths (H) 1999]}$$

**Solution.** Define a function $\phi$ on $[a, a+h]$ as follows :

$$\phi(x) = f(x) + (a+h-x) f'(x) + \tfrac{1}{2}(a+h-x)^2 A, \qquad \ldots(1)$$

where $A$ is a constant to be determined by $\phi(a) = \phi(a+h)$

*i.e.*,
$$f(a) + hf'(a) + \frac{h^2}{2} A = f(a+h). \qquad \ldots(2)$$

Since $f'$ is continuous on $[a, a+h]$, $f$ and $f'$ are continuous on $[a, a+h]$. Further $(a+h-x)$, $(a+h-x)^2$ are also continuous on $[a, a+h]$ and so by (1), $\phi$ is continuous on $[a, a+h]$. Since $f'$ is derivable on $]a, a+h[$, so by (1), $\phi$ is derivable on $]a, a+h[$. Thus $\phi$ satisfies all the conditions of Rolle's Theorem and so there exists some $c \in\ ]a, a+h[$ such that

$$\phi'(c) = 0. \qquad \ldots(3)$$

From (1), $\phi'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - (a+h-x) A$.

$$\therefore \quad 0 = \phi'(c) = (a+h-c)\,[f''(c) - A] \qquad \ldots(4)$$

$\Rightarrow \quad f''(c) - A = 0$, since $c \in\ ]a, a+h[$ means $a + h - c \neq 0$.

Putting $A = f''(c)$ in (2), we obtain

$$f(a+h) = f(a) + hf'(a) + \tfrac{1}{2} h^2 f''(c).$$

**Example 31.** *If a function $f$ is twice differentiable on $[a, a+h]$, then show that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a + \theta h),\ 0 < \theta < 1.$$

**Solution.** Refer to Example 30. We take $c = a + \theta h \in\ ]a, a+h[$, $0 < \theta < 1$ in equation (3) to get $\phi'(a + \theta h) = 0$. From (4), we get

$\phi'(a + \theta h) = (h - \theta h)\,[f''(a + \theta h) - A]$ or $0 = h(1-\theta)\,[f''(a + \theta h) - A]$.

$\therefore \quad f''(a + \theta h) - A = 0$, since $h \neq 0$ and $0 < \theta < 1$.

Putting $A = f''(a + \theta h)$ in (2), we get the desired result.

**Example 32.** *Let $f$ and $g$ be two functions defined and continuous on $[a, b]$ and derivable on $(a, b)$. Show that there exists some $c \in (a, b)$ such that*

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}.$$

**Solution.** We define a function $\phi$ on $[a, b]$ as follows :

$$\phi(x) = \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix} - \left(\frac{x-a}{b-a}\right) \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}. \qquad \ldots(1)$$

By the given hypotheses, $\phi$ is continuous on $[a, b]$ and derivable on $(a, b)$. Also $\phi(a) = 0 = \phi(b)$. Thus $\phi$ satisfies the conditions of Rolle's theorem and so there exists some point $c \in (a, b)$ such that $\phi'(c) = 0$.

From (1), we have

$$\phi'(x) = \begin{vmatrix} f(a) & f'(x) \\ g(a) & g'(x) \end{vmatrix} - \frac{1}{(b-a)} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}.$$

$$\therefore \quad 0 = \phi'(c) = \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix} - \frac{1}{(b-a)} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}.$$

Hence
$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}.$$

## SOME HARDER EXAMPLES

**Example 33.** *If $f''$ be continuous on $[a, b]$ and derivable on $]a, b[$, then prove that*

$$f(b) - f(a) - \tfrac{1}{2}(b-a)\{f'(a) + f'(b)\} = -\frac{(b-a)^3}{12} f'''(d),$$

*for some real number $d$ between $a$ and $b$.*

**Solution.** Define a function $g$ on $[a, b]$ as follows :

$$g(x) = f(x) - f(a) - \tfrac{1}{2}(x-a)\{f'(a) + f'(x)\} + A(x-a)^3,$$

where $A$ is a constant to be suitably chosen.

Since $f''$ is continuous on $[a, b]$, so $f$ and $f'$ are continuous on $[a, b]$. Thus $g$ is continuous on $[a, b]$.

Similarly, $g$ is derivable on $]a, b[$.

Let $A$ be chosen such that $g(a) = g(b)$.

$$\therefore \quad 0 = f(b) - f(a) - \tfrac{1}{2}(b-a)\{f'(a) + f'(b)\} + A(b-a)^3. \qquad \ldots(1)$$

Since the function $g$ satisfies all the conditions of Rolle's theorem in $[a, b]$, therefore, there exists a real number $c$ between $a$ and $b$ such that

$$g'(c) = 0.$$

We have $\quad g'(x) = f'(x) - \tfrac{1}{2}\{f'(a) + f'(x)\} - \tfrac{1}{2}(x-a)f''(x) + 3A(x-a)^2$

or $\quad g'(x) = \tfrac{1}{2}\{f'(x) - f'(a)\} - \tfrac{1}{2}(x-a)f''(x) + 3A(x-a)^2.$

Thus $g'(c) = 0$ implies that

$$\tfrac{1}{2}\{f'(c) - f'(a)\} - \tfrac{1}{2}(c-a)f''(c) + 3A(c-a)^2 = 0. \qquad \text{...(2)}$$

Let $h$ be a function defined on $[a, c]$ as follows :

$$h(x) = \tfrac{1}{2}\{f'(x) - f'(a)\} - \tfrac{1}{2}(x-a)f''(x) + 3A(x-a)^2. \qquad \text{...(3)}$$

Then (*i*) $h$ is continuous on $[a, c]$,

(*ii*) $h$ is derivable on $]a, c[$,

(*iii*) $h(c) = 0$ and $h(a) = 0$, by (2) and (3) ; so that $h(a) = h(c)$.

Since the function $h$ satisfies all the conditions of Rolle's theorem in $[a, c]$, therefore, there exists a real number $d$ $(a < d < c < b)$, such that

$$h'(d) = 0.$$

From (3), $\quad h'(x) = \tfrac{1}{2}f''(x) - \tfrac{1}{2}f''(x) - \tfrac{1}{2}(x-a)f'''(x) + 6A(x-a).$

$$\therefore \quad 0 = h'(d) = -\tfrac{1}{2}(d-a)f'''(d) + 6A(d-a).$$

*i.e.*, $\quad A = f'''(d)/12$, since $(d-a) \neq 0$. ...(4)

From (1) and (4), we have

$$f(b) - f(a) - \tfrac{1}{2}(b-a)\{f'(a) + f'(b)\} = -\tfrac{1}{12}(b-a)^3 f'''(d).$$

**Example 34.** *A function $f$ is such that its second derivative is continuous on $[a, a+h]$ and derivable on $]a, a+h[$, show that there exists a number $\theta$ between 0 and 1 such that*

$$f(a+h) - f(a) - \frac{1}{2}h\{f'(a) + f'(a+h)\} = -\frac{h^3}{12}f'''(a+\theta h).$$

[D.U., Maths (H) 1990]

**Solution.** Taking $b = a + h$ *i.e.*, $h = b - a$ and $d = a + \theta h \in\, ]a, a+h[$, $0 < \theta < 1$ in Example 33, we get the desired result.

**Example 35.** *Assuming that $f''(x)$ exists for all $x$ in $[a, b]$, show that*

$$f(c) - f(a)\frac{b-c}{b-a} - f(b)\frac{c-a}{b-a} - \frac{1}{2}(c-a)(c-b)f''(\xi) = 0, \qquad \text{...(1)}$$

*where $c$ and $\xi$ both lie in $[a, b]$.*

**Solution.** The equation (1) can be rewritten as

$$f(c)(b-a) - f(a)(b-c) - f(b)(c-a) - \tfrac{1}{2}(b-a)(c-a)(c-b)f''(\xi) = 0$$

or $\quad f(a)(c-b) - f(b)(c-a) - f(c)(b-a) - \tfrac{1}{2}(a-b)(b-c)(c-a)f''(\xi) = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} - \frac{1}{2}f''(\xi)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0. \qquad \text{...(2)}$$

We shall prove (2) instead of (1).

The relation (2) helps us to define a function $F(x)$ on $[a, b]$ as follows:

$$F(x) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & x \\ f(a) & f(b) & f(x) \end{vmatrix} - A\begin{vmatrix} 1 & 1 & 1 \\ a & b & x \\ a^2 & b^2 & x^2 \end{vmatrix}, \qquad \text{...(3)}$$

$A$ is a constant such that $F(c) = 0$.

From (3), $\quad F(a) = 0,\ F(b) = 0.$ ...(4)

Thus $F(a) = F(c)$ and $F(c) = F(b)$, $a < c < b$. ...(5)

Since $f''$ exists in $[a, b]$, therefore, $f$ and $f'$ are derivable and con-

s in $[a, b]$ an hence derivable and continuous in each of the intervals

and $[c, b]$. From (3) and (5), it follows that F satisfies the conditions

lle's theorem in each of the intervals $[a, c]$ and $[c, b]$. Consequently,

$$F'(\xi_1) = 0, \text{ for some } \xi_1 \in\, ]a, c[,$$

$$F'(\xi_2) = 0, \text{ for some } \xi_2 \in\, ]c, b[.$$

From (3),

$$F'(x) = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 1 \\ f(a) & f(b) & f'(x) \end{vmatrix} - A\begin{vmatrix} 1 & 1 & 0 \\ a & b & 1 \\ a^2 & b^2 & 2x \end{vmatrix} \qquad \text{...(6)}$$

As stated earlier, $f'$ is derivable and continuous in $[a, b]$. So by (6),

derivable and continuous in $[\xi_1, \xi_2] \subset [a, b]$ and $F'(\xi_1) = F'(\xi_2)$.

$'$ satisfies the conditions of Rolle's theorem in $[\xi_1, \xi_2]$ and so there

$\xi \in\, ]\xi_1, \xi_2[\, \subset [a, b]$ such that

$$F''(\xi) = 0.$$

rom (6), $\quad F''(x) = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ f(a) & f(b) & f''(x) \end{vmatrix} - A\begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ a^2 & b^2 & 2 \end{vmatrix}$

$$= f''(x)(b-a) - 2A(b-a)$$

$$F''(\xi) = 0 \Rightarrow f''(\xi) - 2A = 0 \Rightarrow A = \frac{1}{2}f''(\xi). \qquad (\because a \neq b)$$

bstituting $A = \tfrac{1}{2}f''(\xi)$ in $F(c) = 0$ and using (3), we obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{vmatrix} - \frac{1}{2}f''(\xi)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0,$$

oves (2).

**Example 36.** *If* $f(0)=0$ *and* $f''(x)$ *exists on* $[0, \infty[$, *show that*

$$f'(x)-\frac{f(x)}{x}=\frac{1}{2}xf''(\xi),\ 0<\xi<x, \qquad \text{...(1)}$$

*and deduce that if* $f''(x)$ *is positive for positive values of* $x$, *then* $f(x)/x$ *strictly increases in* $]0, \infty[$. [D.U., Maths (H) 1995]

**Solution.** The relation (1) can be written as

$$f(x)-xf'(x)+\frac{1}{2}x^2f''(\xi)=0.$$

Define a function F as follows :

$$F(x)=f(x)-xf'(x)+\frac{1}{2}Ax^2, \qquad \text{...(2)}$$

where $A$ is a constant such that $F(c)=0$, where $c>0$.

We have $F(0)=f(0)=0$ and $F(c)=0$. $\therefore$ $F(0)=F(c)$.

Thus $F$ satisfies the conditions of Rolle's theorem on $[0, c]$.

Therefore, there exists $\xi$ such that $0<\xi<c$ and $F'(\xi)=0$.

From (2), $\quad F'(x)=-xf''(x)+Ax.$

$\therefore \quad F'(\xi)=0 \Rightarrow A=f''(\xi).$

From (2), $\quad F(c)=0 \Rightarrow f(c)-cf'(c)+\frac{1}{2}Ac^2=0$

$$\Rightarrow \quad f'(c)-\frac{f(c)}{c}=\frac{1}{2}cf''(\xi)\ \forall\ c>0.$$

Hence $\quad f'(x)-\frac{f(x)}{x}=\frac{1}{2}xf''(\xi),\ 0<\xi<x.$

(*ii*) Let $\quad G(x)=f(x)/x$, whenever $x>0$.

$$\therefore \quad G'(x)=\frac{xf'(x)-f(x)}{x^2}=\frac{1}{2}f''(\xi)>0, \text{ as } f''(x)>0\ \forall\ x>0$$

Thus $\quad G'(x)>0\ \forall\ x>0.$ ...(3)

If $x_1$ and $x_2$ be any two positive real numbers such that $x_2>x_1$, then by applying Lagrange's mean value theorem to $G$ in $[x_1, x_2]$, we get

$$G(x_2)-G(x_1)=(x_2-x_1)G'(\theta),\ \theta\in\ ]x_1, x_2[$$

Since $G'(\theta)>0$, by (3), and $x_2-x_1>0$, so

$$G(x_2)-G(x_1)>0 \text{ i.e., } G(x_2)>G(x_1) \text{ for } x_2>x_1>0.$$

Hence $G(x)=f(x)/x$ is strictly increasing in $]0, \infty[$.

**Example 37.** *A twice differentiable function* $f$ *is such that*

$$f(a)=f(b)=0 \text{ and } f(c)>0 \text{ for } a<c<b.$$

*Prove that there is at least one value* $\xi$ *between* $a$ *and* $b$ *for which*

$$f''(\xi)<0.$$

**Solution.** Since $f''$ exists in $[a, b]$, therefore $f$ and $f'$ both exist and are continuous in $[a, b]$ and as well as in $[a, c]$ and $[c, b]$, since $a<c<b$. Applying Lagrange's mean value theorem to $f$ on $[a, c]$ and $[a, b]$, we get

$$\frac{f(c)-f(a)}{c-a}=f'(\xi_1),\ a<\xi_1<c,$$

and
$$\frac{f(b)-f(c)}{b-c}=f'(\xi_2),\ c<\xi_2<b, \text{ respectively.}$$

Since $f(a)=f(b)=0$, the above relations become

$$f'(\xi_1)=\frac{f(c)}{c-a},\ f'(\xi_2)=-\frac{f(c)}{b-c},\ a<\xi_1<c<\xi_2<b. \qquad \text{...(1)}$$

Since $f'$ is continuous and derivable on $[\xi_1, \xi_2]$, so on applying Lagrange's mean value theorem to $f'$ on $[\xi_1, \xi_2]$, we obtain

$$\frac{f'(\xi_2)-f'(\xi_1)}{\xi_2-\xi_1}=f''(\xi), \text{ for } \xi_1<\xi<\xi_2. \qquad \text{...(2)}$$

From (1) and (2), we get

$$f''(\xi)=-\frac{f(c)}{\xi_2-\xi_1}\left(\frac{1}{b-c}+\frac{1}{c-a}\right)$$

$$=\frac{-(b-a)f(c)}{(\xi_2-\xi_1)(b-c)(c-a)}<0,$$

since $f(c)>0$, $a<c<b$ and $\xi_1<\xi_2$.

Hence $f''(\xi)<0$ for some $\xi$, $a<\xi<b$.

**Example 38.** *If* $f''$ *be defined on* $[a, b]$ *and if* $|f''(x)|\le M$ *for all* $x$ *in* $[a, b]$, *then prove that*

$$|f(b)-f(a)-\tfrac{1}{2}(b-a)\{f'(a)+f'(b)\}|\le\tfrac{1}{2}(b-a)^2M.$$

**Solution.** Define a function $\phi$ on $[a, b]$ as follows :

$$\phi(x)=f(x)-f(a)-\tfrac{1}{2}(x-a)\{f'(a)+f'(x)\}+A(x-a)^2, \qquad \text{...(1)}$$

where $A$ is a constant to be chosen such that $\quad \phi(b)=0$ ...(2)

*i.e.*, $\quad f(b)-f(a)-\tfrac{1}{2}(b-a)\{f'(a)+f'(b)\}+A(b-a)^2=0.$ ...(3)

Since $f''$ is defined on $[a, b]$, so $f$ and $f'$ are both derivable on $[a, b]$ *i.e.*, $\phi$ is derivable on $[a, b]$ and so $\phi$ is continuous on $[a, b]$.

From (1) and (2), $\phi(a)=0$, $\phi(b)=0$ and so $\phi(a)=\phi(b)$.

By Rolle's Theorem, there exists some $c\in\ ]a, b[$ such that

$$\phi'(c)=0.$$

From (1), $\phi'(x)=f'(x)-\tfrac{1}{2}\{f'(a)+f'(x)\}-\tfrac{1}{2}(x-a)f''(x)+2A(x-a)$

$\therefore \quad \phi'(c)=0 \Rightarrow f'(c)-f'(a)-(c-a)f''(c)+4A(c-a)=0.$ ...(4)

Applying Lagrange's Mean Value Theorem to $f'$ on $[a, c]$, we obtain

$$f'(c)-f'(a)=(c-a)f''(d) \text{ for some } d\in\ ]a, c[.$$

Substituting in (4), we obtain

$$f''(d)-f''(c)+4A=0 \Rightarrow A=\tfrac{1}{4}[f''(c)-f''(d)].$$

Using (1), it follows that $f'(\xi_1) = f'(\xi_2)$. Further,

(*i*) $f'$ is continuous in $[\xi_1, \xi_2]$, (*ii*) $f'$ is derivable in $]\xi_1, \xi_2[$ and (*iii*) $f'(\xi_1) = f'(\xi_2)$. So by Rolle's Theorem as applied to $f'$ in $[\xi_1, \xi_2]$, we obtain

$$f''(\xi) = 0 \text{ for some } \xi \in \,]\xi_1, \xi_2[\, \subset \,]a, b[$$

**Example 41.** *If* $f''(x) > 0$ *for all* $x \in \mathbf{R}$, *then show that*

$$f[\tfrac{1}{2}(x_1 + x_2)] \le \tfrac{1}{2}[f(x_1) + f(x_2)], \qquad \text{...(1)}$$

*for every pair of real numbers* $x_1$ *and* $x_2$. [D.U., Maths (H) 1998]

**Solution.** Let $x_1, x_2 \in \mathbf{R}$ be arbitrary. If $x_1 = x_2$, then the given relation (1) becomes equality and so the result follows. We may suppose that $x_1 < x_2$. Since $f''(x)$ exists $\forall\, x \in \mathbf{R}$, $f$ satisfies the conditions of Lagrange's mean value theorem in each of the intervals $[x_1, \tfrac{1}{2}(x_1 + x_2)]$ and $[\tfrac{1}{2}(x_1 + x_2), x_2]$. Consequently,

$$f[\tfrac{1}{2}(x_1 + x_2)] - f(x_1) = [\tfrac{1}{2}(x_1 + x_2) - x_1]\, f'(c_1),\; x_1 < c_1 < \tfrac{1}{2}(x_1 + x_2)$$

and $$f(x_2) - f[\tfrac{1}{2}(x_1 + x_2)] = [x_2 - \tfrac{1}{2}(x_1 + x_2)]\, f'(c_2),\; \tfrac{1}{2}(x_1 + x_2) < c_2 < x_2 .$$

On subtracting the corresponding sides of these equations, we get

$$2f[\tfrac{1}{2}(x_1 + x_2)] - f(x_1) - f(x_2) = \frac{1}{2}(x_2 - x_1)\,[f'(c_1) - f'(c_2)]. \qquad \text{...(2)}$$

Since $f''(x)$ exists $\forall\, x \in \mathbf{R}$, $f'$ satisfies the conditions of Lagrange's mean value theorem in the interval $[c_1, c_2]$ and so

$$f'(c_2) - f'(c_1) = (c_2 - c_1) f''(t), \text{ for some } t \in \,]c_1, c_2[.$$

As given $f''(t) > 0$ and $c_2 > c_1$, so $(c_2 - c_1) f''(t) > 0$.

Thus $f'(c_2) - f'(c_1) > 0$ *i.e.*, $f'(c_1) - f'(c_2) < 0$.

Also $x_2 > x_1 \Rightarrow x_2 - x_1 > 0$.

Using these inequalities in (2), we get

$$2f[\tfrac{1}{2}(x_1 + x_2)] - f(x_1) - f(x_2) < 0.$$

Hence $f[\tfrac{1}{2}(x_1 + x_2)] < \tfrac{1}{2}[f(x_1) + f(x_2)]$.

**Example 42.** *If* $f', g'$ *are continuous on* $[a - h, a + h]$

## EXAMPLES

**Example 43.** *Show that the function $f$ defined on* **R** *by*

$$f(x) = x^3 - 6x^2 + 12x - 4 \text{ for all } x \in \mathbf{R}$$

*is increasing in every interval.*

**Solution.** Let $f(x) = x^3 - 6x^2 + 12x - 4$.

$$\therefore \quad f'(x) = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4)$$

or $$f'(x) = 3(x-2)^2 \quad \forall \quad x \in \mathbf{R}$$

$$\Rightarrow \quad f'(x) > 0 \text{ when } \quad x \neq 2 \text{ and } f'(2) = 0.$$

Consider any interval $[c, 2]$ ; $c \in \mathbf{R}$.

Obviously, $f$ is continuous in $[c, 2]$ and $f'(x) > 0$ for all $x$ in $]c, 2[$. By Theorem 1, $f$ is strictly increasing in $[c, 2]$.

Similarly, $f$ is strictly increasing in $[2, d]$, $d \in \mathbf{R}$.

**Example 44.** (*a*) *Show that* $x^3 - 6x^2 + 15x + 3 > 0 \ \forall \ x > 0$.

(*b*) *Given* $f(x) = \sin^2 x$, $x \in [0, \pi]$, *on what subintervals of* $[0, \pi]$ *is* $f$ *increasing and decreasing ?*

**Solution.** (*a*) Let $f(x) = x^3 - 6x^2 + 15x + 3$.

$\therefore \quad f'(x) = 3x^2 - 12x + 15 = 3(x^2 - 4x + 5)$

$\Rightarrow \quad f'(x) = 3[(x-2)^2 + 1]$

$\Rightarrow \quad f'(x) > 0 \quad \forall \ x \in \mathbf{R}$.

In particular, $f$ is strictly increasing $\forall \ x > 0$

$\Rightarrow \quad f(x) > f(0) \ \ \forall \ x > 0$, where $f(0) = 3 > 0$.

Hence $\quad f(x) > 0 \quad \forall \ x > 0$.

(*b*) $\quad f(x) = \sin^2 x \ \Rightarrow \ f'(x) = 2 \sin x \cos x = \sin 2x$.

$f(x)$ is an increasing function if $f'(x) > 0 \Rightarrow \sin 2x > 0$

$\Rightarrow \quad 0 < 2x < \pi \Rightarrow 0 < x < \pi/2$.

Thus $f(x)$ is increasing in the subinterval $[0, \pi/2]$.

Now $f(x)$ is decreasing function if $f'(x) < 0 \Rightarrow \sin 2x < 0$

$\Rightarrow \quad \pi < 2x < 2\pi \Rightarrow \pi/2 < x < \pi$.

Hence $f(x)$ is decreasing in the subinterval $[\pi/2, \pi]$.

**Example 45.** *Show that the function* $f(x) = x^5 - 5x^3 + 100x - 90$ *is increasing in every interval.* [D.U., B.C.A. 1998]

**Solution.** We have $f'(x) = 5x^4 - 15x^2 + 100 = 5(x^4 - 3x^2 + 20)$

$\therefore \quad f'(x) = 5[(x^2 - 2)^2 + x^2 + 16] > 0 \ \forall \ x \in \mathbf{R}$.

Hence $f$ is increasing $\forall \ x \in \mathbf{R}$ and so $f$ is increasing in every subinterval.

**Example 46.** *Separate the intervals in which the function*

$$f(x) = 2x^3 - 15x^2 + 36x - 1 \ \forall \ x \in \mathbf{R}$$

*is increasing or decreasing.*

**Solution.** $f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3)$.

We observe that

$$f'(x) \geq 0 \text{ for } x \geq 3 \text{ or } x \leq 2 \text{ and } f'(x) \leq 0 \text{ for } 2 \leq x \leq 3.$$

Hence $f$ is increasing in $]-\infty, 2]$ and $[3, \infty[$ and decreasing in $[2, 3]$.

**Example 47.** *Show that*

(*a*) $\dfrac{x}{1+x} < \log(1+x) < x, x > 0.$

(*b*) $x - \dfrac{x^2}{2} < \log(1+x) < x - \dfrac{x^2}{2(1+x)}, x > 0.$

[D.U., B.Sc. (G) 1996 ; B.C.A. 1998 ; B.A. (P) 1990]

(*c*) $\dfrac{x^2}{2(1+x)} < x - \log(1+x) < \dfrac{x^2}{2}, x > 0.$

[D.U., Physics (H) 1999 ; B.A. (P) 1991]

**Solution.** Let $f(x) = \log(1+x) - \dfrac{x}{1+x}$, so that $f(0) = 0$.

We shall show that $f(x) > 0$ for $x > 0$.

Now $\quad f'(x) = \dfrac{1}{1+x} - \dfrac{1}{(1+x)^2} = \dfrac{x}{(1+x)^2} > 0$, for $x > 0$.

Thus $f(x)$ is an increasing function $\forall\, x > 0$

$\Rightarrow \quad f(x) > f(0)$, for $x > 0$

$\Rightarrow \quad f(x) > 0$, for $x > 0 \qquad (\because\ f(0) = 0)$

$\Rightarrow \quad \log(1+x) - \dfrac{x}{1+x} > 0, x > 0$

$\therefore \quad \dfrac{x}{1+x} < \log(1+x), x > 0.$

Let $\quad \phi(x) = x - \log(1+x)$, so that $\phi(0) = 0$.

$\therefore \quad \phi'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0$, for $x > 0$.

Thus $\phi(x)$ is an increasing function of $x$, for $x > 0$

$\Rightarrow \quad \phi(x) > \phi(0)$, for $x > 0$

$\Rightarrow \quad \phi(x) > 0 \qquad (\because\ \phi(0) = 0)$

$\Rightarrow \quad x - \log(1+x) > 0$, for $x > 0$

$\Rightarrow \quad \log(1+x) < x$, for $x > 0$.

Hence $\dfrac{x}{1+x} < \log(1+x) < x, x > 0.$

(*b*) Let $f(x) = \log(1+x) - \left(x - \dfrac{x^2}{2}\right), f(0) = 0.$

$\therefore \quad f'(x) = \dfrac{1}{1+x} - 1 + x = \dfrac{x^2}{1+x} > 0. \qquad (\because\ x > 0)$

$\Rightarrow f(x)$ is an increasing function of $x$, for $x > 0$

$\Rightarrow f(x) > f(0)$, for $x > 0 \Rightarrow f(x) > 0 \qquad (\because\ f(0) = 0) \qquad \ldots(2)$

$\Rightarrow \log(1+x) - \left(x - \dfrac{x^2}{2}\right) > 0$, for $x > 0$

$\therefore \quad x - \dfrac{x^2}{2} < \log(1+x), x > 0.$

Let $\quad \phi(x) = x - \dfrac{x^2}{2(1+x)} - \log(1+x)$, so that $\phi(0) = 0$.

$$\therefore \quad \phi'(x) = 1 - \frac{1}{2}\left\{\frac{2x + x^2}{(1+x)^2}\right\} - \frac{1}{1+x}$$

$$= \frac{x^2}{2(1+x)^2} > 0 \qquad (\because\ x > 0)$$

$\Rightarrow \quad \phi(x)$ is an increasing function, for $x > 0$

$\Rightarrow \quad \phi(x) > \phi(0)$, for $x > 0$

$\Rightarrow \quad \phi(x) > 0 \qquad (\because\ \phi(0) = 0)$

$\Rightarrow \quad x - \dfrac{x^2}{2(1+x)} - \log(1+x) > 0$, for $x > 0$

$\therefore \quad \log(1+x) < x - \dfrac{x^2}{2(1+x)}$, for $x > 0$.

Hence $\quad x - \dfrac{x^2}{2} < \log(1+x) < x - \dfrac{x^2}{2(1+x)}$, for $x > 0$.

(*c*) By part (*b*), we have

$$-x + \frac{x^2}{2} > -\log(1+x) > \frac{x^2}{2(1+x)} - x, \; x > 0$$

or

$$\frac{x^2}{2} > x - \log(1+x) > \frac{x^2}{2(1+x)}, \; x > 0.$$

Hence $\quad \dfrac{x^2}{2(1+x)} < x - \log(1+x) < \dfrac{x^2}{2}, x > 0.$

**Example 48.** *Prove that* $x < \sin^{-1} x < \dfrac{x}{\sqrt{1-x^2}}$, *if* $0 < x < 1$.

[D.U., B.Sc. (G) 1993]

**Solution.** Let $f(x) = \sin^{-1} x - x$, so that $f(0) = 0$.

$$\therefore \quad f'(x) = \frac{1}{\sqrt{1-x^2}} - 1 = \frac{1 - \sqrt{1-x^2}}{\sqrt{1-x^2}}$$

$\Rightarrow f'(x) > 0$, for $0 < x < 1$

$\Rightarrow f$ is strictly increasing in $0 < x < 1$

$\Rightarrow f(x) > f(0)$, for $0 < x < 1$ $\quad (\because x > 0)$

$\Rightarrow f(x) > 0$, for $0 < x < 1$.

$\Rightarrow \sin^{-1} x - x > 0$, for $0 < x < 1$

$\Rightarrow \sin^{-1} x > x$, for $0 < x < 1$

$\Rightarrow x < \sin^{-1} x$, for $0 < x < 1$.

Let $\quad g(x) = \dfrac{x}{\sqrt{1-x^2}} - \sin^{-1} x$, so that $g(0) = 0$.

$$\therefore \quad g'(x) = \frac{\sqrt{1-x^2} + \dfrac{x(-2x)}{2\sqrt{1-x^2}}}{(1-x^2)} - \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{(1-x^2)\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = \frac{x^2}{(1-x^2)^{3/2}}$$

$\Rightarrow g'(x) >$, for $0 < x < 1$

$\Rightarrow g$ is strictly increasing, for $0 < x < 1$

$\Rightarrow g(x) > g(0)$, for $0 < x < 1$ $\quad (\because x > 0)$

$\Rightarrow g(x) > 0$, for $0 < x < 1$

$\Rightarrow \dfrac{x}{\sqrt{1-x^2}} > \sin^{-1} x$, for $0 < x < 1$

$\Rightarrow \sin^{-1} x < \dfrac{x}{\sqrt{1-x^2}}$, for $0 < x < 1$.

**Example 49.** *Using Lagrange's mean value theorem, show that*

$$\frac{x}{1+x} < \log(1+x) < x, \ x > 0. \qquad \text{[D.U., B.Sc. (G) 1992]}$$

**Solution.** Let $f(x) = \log(1+x)$ in $[0, x]$, so that $f'(x) = \dfrac{1}{1+x}$.

Since $f$ is continuous in $[0, x]$ and derivable in $]0, x[$, so by Lagrange's mean value theorem, there exists some $\theta$, $0 < \theta < 1$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\theta x) \qquad (\because \ ]0, x[ = ]0x, 1x[, c = \theta x)$$

or $$\log(1+x) = \frac{x}{1+\theta x}. \qquad [\because f(0) = 0] \quad \ldots(1)$$

Now $\quad 0 < \theta < 1$ and $x > 0 \Rightarrow \theta x < x$

$$\Rightarrow \quad 1 + \theta x < 1 + x \Rightarrow \frac{1}{1+\theta x} > \frac{1}{1+x}$$

$$\Rightarrow \quad \frac{x}{1+\theta x} > \frac{x}{1+x} \Rightarrow \frac{x}{1+x} < \frac{x}{1+\theta x}. \quad \ldots(2)$$

Again $\quad 0 < \theta < 1$ and $x > 0 \Rightarrow 1 < 1 + \theta x$

$$\Rightarrow \quad \frac{1}{1+\theta x} < 1 \Rightarrow \frac{x}{1+\theta x} < x. \quad \ldots(3)$$

From (2) and (3), we obtain

$$\frac{x}{1+x} < \frac{x}{1+\theta x} < x. \quad \ldots(4)$$

From (1) and (4), we obtain

$$\frac{x}{1+x} < \log(1+x) < x.$$

**Example 50.** *Applying Lagrange's mean value theorem to the function defined by* $f(x) = \log(1+x)$ *for all* $x > 0$, *show that*

$$0 < [\log(1+x)]^{-1} - x^{-1} < 1 \text{ whenever } x > 0.$$

[D.U., Maths (H) 1996, 93 ; B.Sc. (G) 1994, 89 ;
B.C.A. 1999; B.A. (P) 1998]

**Solution.** First proceed like Example 49 to obtain

$$\frac{x}{1+x} < \log(1+x) < x$$

$$\Rightarrow \quad \frac{1+x}{x} > \frac{1}{\log(1+x)} > \frac{1}{x}$$

$$\Rightarrow \quad 1 + \frac{1}{x} > \frac{1}{\log(1+x)} > \frac{1}{x}$$

$$\Rightarrow \quad 1 > \frac{1}{\log(1+x)} - \frac{1}{x} > 0.$$

Hence $\quad 0 < [\log(1+x)]^{-1} - x^{-1} < 1, x > 0.$

**Example 51.** Use the mean value theorem to prove

$$\frac{x}{1+x^2} < \tan^{-1} x < x, \text{ if } x > 0. \qquad \text{[D.U., Maths (H) 1988]}$$

**Solution.** Let $\quad f(x) = \tan^{-1} x$ in $[0, x]$.

Then $f$ satisfies the conditions of Lagrange's mean value theorem in $[0, x]$. Consequently, there exists some $\theta$ satisfying $0 < \theta < 1$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\theta x) \left[ f'(x) = \frac{1}{1+x^2} \text{ and } f(0) = 0 \right]$$

or $$\tan^{-1} x = \frac{x}{1+\theta^2 x^2}. \quad \ldots(1)$$

Now $\quad 0 < \theta < 1$ and $x > 0 \Rightarrow \theta^2 x^2 < x^2 \Rightarrow 1 + \theta^2 x^2 < 1 + x^2$

$$\Rightarrow \quad \frac{x}{1+\theta^2 x^2} > \frac{x}{1+x^2} \Rightarrow \frac{x}{1+x^2} < \frac{x}{1+\theta^2 x^2}. \quad \ldots(2)$$

Again $\quad 0 < \theta < 1$ and $x > 0 \Rightarrow 1 < 1 + \theta^2 x^2$

$$\Rightarrow \quad \frac{1}{1+\theta^2 x^2} < 1 \Rightarrow \frac{x}{1+\theta^2 x^2} < x. \quad \ldots(3)$$

From (2) and (3), we obtain

$$\frac{x}{1+x^2} < \frac{x}{1+\theta^2 x^2} < x, \; x > 0. \qquad \ldots(4)$$

From (1) and (4), we obtain

$$\frac{x}{1+x^2} < \tan^{-1} x < x, \; x > 0.$$

**Example 52.** *Show that*

$$\frac{1}{2}x^2 < x - \log(1+x) < \frac{1}{2}\frac{x^2}{(1+x)}, \; \textit{if} \; -1 < x < 0.$$

[D.U., Maths (H) 1997]

**Solution.** Let $f(x) = x - \log(1+x) - \frac{1}{2}x^2$, so that $f(0) = 0$.

$$\therefore \qquad f'(x) = 1 - \frac{1}{1+x} - x = -\left(\frac{x^2}{1+x}\right). \qquad \ldots(1)$$

Now $\quad -1 < x < 0 \Rightarrow x + 1 > 0$. Thus, by (1),

$f'(x) < 0$, for $-1 < x < 0$

$\Rightarrow$ $f$ is decreasing, for $-1 < x < 0$

$\Rightarrow$ $f(x) > f(0)$, for $-1 < x < 0$

$\Rightarrow$ $f(x) > 0$, for $-1 < x < 0$

$\Rightarrow$ $\frac{1}{2}x^2 < x - \log(1+x)$, for $-1 < x < 0$.

Let $\quad g(x) = \frac{1}{2}\frac{x^2}{(1+x)} - x + \log(1+x)$, so that $g(0) = 0$.

$$\therefore \quad g'(x) = \frac{2x + x^2}{2(1+x)^2} - 1 + \frac{1}{1+x} = -\left\{\frac{x^2}{2(1+x)^2}\right\}$$

$\Rightarrow$ $g'(x) < 0$, for $-1 < x < 0$

$\Rightarrow$ $g$ is decreasing, for $-1 < x < 0$

$\Rightarrow$ $g(x) > g(0)$, for $-1 < x < 0$

$\Rightarrow$ $g(x) > 0$, for $-1 < x < 0$

$\Rightarrow$ $x - \log(1+x) < \frac{1}{2}\frac{x^2}{(1+x)}, \; -1 < x < 0$.

**Example 53.** *Prove that*

$$x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}, \; \textit{if} \; x > 0.$$

[D.U., Physics (H) 1996]

**Solution.** Let $f(x) = \log(1+x) - \left[x - \frac{x^2}{2} + \frac{x^3}{3(1+x)}\right]$.

$$\therefore \quad f'(x) = \frac{1}{1+x} - 1 + x - \frac{3x^2 + 2x^3}{3(1+x)^2} = \frac{x^3}{3(1+x)^2}$$

---

$\Rightarrow$ $f'(x) > 0$, for $x > 0$

$\Rightarrow$ $f$ is increasing, for $x > 0$

$\Rightarrow$ $f(x) > f(0)$, for $x > 0$

$\Rightarrow$ $f(x) > 0$ $\qquad (\because \; f(0) = 0)$

$\Rightarrow$ $x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x)$, if $x > 0$.

Let $g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \log(1+x)$.

$$\therefore \quad g'(x) = 1 - x + x^2 - \frac{1}{1+x} = \frac{x^3}{1+x}$$

$\Rightarrow$ $g'(x) > 0$, for $x > 0$

$\Rightarrow$ $g$ is increasing, for $x > 0$

$\Rightarrow$ $g(x) > g(0)$, for $x > 0$

$\Rightarrow$ $g(x) > 0$ $\qquad (\because \; g(0) = 0)$

$\Rightarrow$ $\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$, if $x > 0$.

**Example 54.** *Show that*

$$\frac{x}{1+x} < \log(1+x) < x \quad \textit{for} \; x > -1, x \neq 0.$$

[D.U., B.Sc. (G) 1997, 95]

**Solution.** Refer to Example 47 (*a*).

The given inequalities are true for $x > 0$.

Now we shall establish the inequalities for $-1 < x < 0$.

Let $\quad f(x) = \log(1+x) - \frac{x}{1+x} \Rightarrow f'(x) = \frac{x}{(1+x)^2}$,

and $\quad \varphi(x) = x - \log(1+x) \Rightarrow \varphi'(x) = \frac{x}{1+x}$.

Now $-1 < x < 0 \Rightarrow 0 < 1 + x < 1$ and $x < 0$. It now follows that

$f'(x) < 0$ and $\varphi'(x) < 0$, for $-1 < x < 0$

$\Rightarrow$ $f(x)$ and $\varphi(x)$ are decreasing functions, for $-1 < x < 0$

$\Rightarrow$ $f(x) > f(0) = 0$ and $\varphi(x) > \varphi(0) = 0$, for $-1 < x < 0$.

$\therefore$ $f(x) > 0$ and $\varphi(x) > 0$, for $-1 < x < 0$.

Hence $\quad \frac{x}{1+x} < \log(1+x) < x$, for $-1 < x < 0$.

**Example 55.** *If* $0 < x < 1$, *show that*

$$2x < \log\frac{1+x}{1-x} < 2x\left(1 + \frac{1}{3}\cdot\frac{x^2}{1-x^2}\right).$$

**Solution.** Let $f(x) = \log\frac{1+x}{1-x} - 2x$.

$$\therefore \quad f'(x) = \frac{1-x}{1+x}\left\{\frac{(1-x)+(1+x)}{(1-x)^2}\right\} - 2 = \frac{2}{1-x^2} - 2 = \frac{2x^2}{1-x^2}.$$

Clearly, $f'(x) > 0$, for $0 < x < 1 \Rightarrow f$ is increasing, for $0 < x < 1$.
In particular, $f(x) > f(0) = 0$ as $x > 0$

$$\therefore \quad \log\frac{1+x}{1-x} > 2x, \text{ for } 0 < x < 1.$$

**Hence** $\quad 2x < \log\dfrac{1+x}{1-x}$, for $0 < x < 1$.

**Let** $\quad g(x) = 2x\left(1 + \dfrac{1}{3}\cdot\dfrac{x^2}{1-x^2}\right) - \log\dfrac{1+x}{1-x}.$

$$\therefore \quad g'(x) = 2 + \frac{2}{3}\left[\frac{(1-x^2).3x^2 + x^3.2x}{(1-x^2)^2}\right] - \frac{2}{1-x^2}$$

$$= \frac{6(1-x^2)^2 + 2(3x^2 - x^4) - 6(1-x^2)}{3(1-x^2)^2} = \frac{4x^4}{3(1-x^2)^2} > 0, \text{ if } 0 < x < 1$$

Thus $g(x)$ is increasing, for $0 < x < 1$. In particular,

$$g(x) > g(0) = 0 \quad \Rightarrow g(x) > 0.$$

**Hence** $\quad \log\dfrac{1+x}{1-x} < 2x\left(1 + \dfrac{1}{3}\cdot\dfrac{x^2}{1-x^2}\right).$

**Example 56.** *Prove that* $\tan x > x$, *whenever* $0 < x < \pi/2$.

**Solution.** Let $c$ be any real number such that $0 < c < \pi/2$.

Let $\quad f(x) = \tan x - x$ for all $x \in [0, c]$.

Then $f$ is continuous as well as derivable on $[0, c]$.

Now $f'(x) = \sec^2 x - 1 = \tan^2 x > 0$, for $0 < x < c$.

Thus $f$ is strictly increasing in $[0, c]$

$\Rightarrow \quad f(c) > f(0)$, for $c > 0$.

But $f(0) = 0$. Therefore $f(c) > 0 \Rightarrow \tan c - c > 0$.

Since $c$ is any real number such that $0 < c < \pi/2$, therefore,

$$\tan x > x, \text{ whenever } 0 < x < \pi/2.$$

**Example 57.** *Show that* $\dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1, 0 < x < \dfrac{\pi}{2}$.

**Solution.** Let $f$ be defined as

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases} \qquad \text{...(1)}$$

Then $f$ is continuous in $[0, \pi/2]$ and derivable in $]0, \pi/2[$.

We have $\quad f'(x) = \dfrac{x\cos x - \sin x}{x^2}. \qquad$ ...(2)

Let $\quad g(x) = x\cos x - \sin x$ in $[0, \pi/2]$.

$\therefore \quad g'(x) = \cos x - x\sin x - \cos x$

$\qquad = -x\sin x < 0 \quad$ in $]0, \pi/2[$

$\Rightarrow \quad g$ is strictly decreasing in $[0, \pi/2]$.

$\Rightarrow \quad g(x) < g(0) = 0$ in $[0, \pi/2] \qquad (\because x > 0)$

Using in (2), it follows that $f'(x) < 0$ in $]0, \pi/2[$

$\Rightarrow \quad f$ is strictly decreasing in $[0, \pi/2]$.

$\Rightarrow \quad f(0) > f(x) > f(\pi/2)$, for $0 < x < \pi/2$

$\Rightarrow \quad 1 > \dfrac{\sin x}{x} > \dfrac{1}{\pi/2}$, using (1)

**Hence** $\quad \dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1$, for $0 < x < \dfrac{\pi}{2}$.

**Example 58.** *Prove that*

$$\frac{\tan x}{x} > \frac{x}{\sin x}, \textit{ whenever } 0 < x < \frac{\pi}{2}.$$

(D.U., Maths (H) 1999 ; B.Sc. (G) 1991 ; B.A. (P) 1992)

**Solution.** $\quad \dfrac{\tan x}{x} - \dfrac{x}{\sin x} = \dfrac{\tan x\sin x - x^2}{x\sin x}.$

Since $x\sin x > 0$ for all $x$ in $]0, \pi/2[$, therefore, we need to show that

$$\tan x\sin x - x^2 > 0, \forall x \in ]0, \pi/2[.$$

Let $c$ any real number in $]0, \pi/2[$.

Let $\quad f(x) = \tan x\sin x - x^2 \;\; \forall \;\; x \in [0, c]$.

Then $f$ is continuous as well as derivable in $[0, c]$.

Now $\quad f'(x) = \sec^2 x\sin x + \tan x\cos x - 2x$

$\qquad = \sin x(\sec^2 x + 1) - 2x.$

The form of $f'(x)$ is such that we cannot decide about its sign.

Let $\quad g(x) = f'(x)$ for all $x$ in $[0, c]$.

The function $g$ is continuous as well as derivable on $[0, c]$.

Now $\quad g'(x) = \cos x(\sec^2 x + 1) + \sin x . 2\sec^2 x\tan x - 2$

$\qquad = (\sqrt{\sec x} - \sqrt{\cos x})^2 + 2\sin^2 x\sec^3 x.$

Since $g'(x) > 0$ for all $x \in \;]0, c[ \Rightarrow g$ is strictly increasing in $[0, c]$
$\Rightarrow g(x) > g(0)$, whenever $0 < x < c$. **Since** $g(0) = 0$, this means that $g(x) > 0$, whenever $0 < x < c \Rightarrow f'(x) > 0$, whenever $0 < x < c \Rightarrow f$ is strictly increasing in $[0, c] \Rightarrow f(c) > f(0) = 0 \Rightarrow f(c) > 0$

$\Rightarrow \quad \tan c\sin c - c^2 > 0$

$$\Rightarrow \quad \frac{\tan c}{c} - \frac{c}{\sin c} > 0.$$

Since $c$ is any point of $]0, \pi/2[$, it follows that

$$\frac{\tan x}{x} > \frac{x}{\sin x}, \quad \text{whenever} \quad 0 < x < \frac{\pi}{2}.$$

## EXERCISES

Prove the following inequalities :

1. $1 - x < e^{-x} < 1 - x + \frac{1}{2}x^2$, if $x > 0$.
2. $\cos x + \sin x > 1$, for $0 < x < \frac{1}{2}\pi$.
3. $x < -\log(1-x) < x(1-x)^{-1}$, if $0 < x < 1$.
4. $\frac{x}{1+x^2} < \tan^{-1} x < x$, for $x > 0$.
5. Show that $1 - x < -\log x < \frac{1}{x} - 1$, if $0 < x < 1$.
6. Show that $\sin x < x$, for $0 < x < \pi/2$.
7. Show that $\tan x > x > \sin x$, for $0 < x < \pi/2$.
8. Use the mean value theorem to establish the following :

$$e^a(x-a) < e^x - e^a < e^x(x-a), \text{ if } a < x.$$

[**Hint.** Apply LMV Theorem to $f(x) = e^x$ in $[a, x]$, to get $\frac{e^x - e^a}{x - a} = e^c$ for some $c \in \,]a, x[$. Now $a < c < x \Rightarrow e^a < e^c < e^x$]

9. Use the mean value theorem to establish

$$(1+x)^{1/2} < 1 + \frac{1}{2}x, \text{ if } -1 < x < 0 \text{ or } x > 0.$$

[**Hint.** Apply LMV Theorem to $f(x) = \sqrt{1+x} - 1$ in $[0, x]$ to get

$$\frac{f(x) - f(0)}{x - 0} = f'(c) = \frac{1}{2\sqrt{1+c}}, \quad \text{for some } c \in \,]0, x[$$

or $\quad \frac{\sqrt{1+x}-1}{x} < \frac{1}{2}$ [$\because \sqrt{1+c} > 1$]. Hence $\sqrt{1+x} < 1 + \frac{1}{2}x$, as $x > 0$.

Now apply LMV Theorem to $f(x)$ in $[x, 0]$, where $x > -1$.

$$\therefore \quad \frac{f(0) - f(x)}{0 - x} = \frac{1}{2\sqrt{1+d}}, \text{ for some } d \in \,]x, 0[$$

$$\text{or} \quad \frac{\sqrt{1+x}-1}{x} = \frac{1}{2\sqrt{1+d}}.$$

Now $\quad x < d < 0 \Rightarrow \sqrt{1+x} < \sqrt{1+d} < 1$ ...(1)

$$\Rightarrow \frac{\sqrt{1+x}-1}{x} > \frac{1}{2} \Rightarrow \sqrt{1+x} - 1 < \frac{1}{2}x, \text{ as } x < 0]$$

---

10. Use the mean value theorem to show that

$$1 + \frac{x}{2\sqrt{1+x}} < \sqrt{1+x} < 1 + \frac{x}{2}, \text{ for } -1 < x < 0.$$

[**Hint.** From (1) of Ex. 9, $\quad \frac{1}{2} < \frac{1}{2\sqrt{1+d}} < \frac{1}{2\sqrt{1+x}}$

$$\text{or} \quad \frac{1}{2} < \frac{\sqrt{1+x}-1}{x} < \frac{1}{2\sqrt{1+x}}, \text{ by (1)}$$

$$\text{or} \quad \frac{1}{2}x > \sqrt{1+x} - 1 > \frac{x}{2\sqrt{1+x}}, \text{ as } x < 0]$$

## 7.6 Taylor's Theorem

### 7.6.1 Taylor's Theorem with Lagrange's Remainder

*If a function f defined on $[a, a+h]$ is such that*

*(i) the $(n-1)$th derivative $f^{n-1}$ is continuous in $[a, a+h]$,*

*(ii) the $(n-1)$th derivative $f^{n-1}$ is derivable in $]a, a+h[$,*

*then there exists some $\theta, 0 < \theta < 1$, such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n$$

*where* $\quad R_n = \frac{h^n}{n!}f^n(a + \theta h). \qquad$ **[Lagrange's Remainder]**

**Proof.** By the given hypotheses; $f, f', f'', \ldots, f^{n-1}$ are all continuous in $[a, a+h]$ and derivable in $]a, a+h[$. ...(1)

We define a function $\phi$ on $[a, a+h]$ as follows :

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \ldots$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + A(a+h-x)^n \; \forall \; x \in [a, a+h], \quad \ldots(2)$$

where $A$ is constant to be determined by $\phi(a) = \phi(a+h)$. ...(3)

From (2) and (3), we obtain

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + Ah^n = f(a+h). \quad \ldots(4)$$

From (1) and (2), we see that

$\phi$ is continuous in $[a, a+h]$ and $\phi$ is derivable in $]a, a+h[$.

Also $\phi(a) = \phi(a+h)$.

Thus $\phi$ satisfies all the conditions of Rolle's Theorem and so there exists some $\theta, 0 < \theta < 1$. such that

$$\phi'(a + \theta h) = 0. \quad \ldots(5) \; \{c = a + \theta h \in \,]a, a+h[\,\}$$

Differentiating (2) on both sides, we obtain

$$\phi'(x) = f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x) + \ldots$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x) - nA(a+h-x)^{n-1}$$

$$\therefore \quad \phi'(h+\theta h) = \frac{(h-\theta h)^{n-1}}{(n-1)!} f^n(a+\theta h) - nA(h-\theta h)^{n-1}$$

or
$$0 = h^{n-1}(1-\theta)^{n-1}\left[\frac{1}{(n-1)!} f^n(a+\theta h) - nA\right] \quad (h \neq 0,\ \theta \neq 1)$$

$\therefore \quad A = \frac{1}{n!} f^n(a+\theta h)$. Putting this value of $A$ in (4), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h).$$

### 7.6.2 Taylor's Theorem with Cauchy's Remainder

*If a function f defined on $[a, a+h]$ is such that*

*(i) the $(n-1)$th derivative $f^{n-1}$ is continuous in $[a, a+h]$,*

*(ii) the $(n-1)$th derivative $f^{n-1}$ is derivable in $]a, a+h[$,*

*then there exists some $\theta$, $0 < \theta < 1$ such that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n,$$

*where* $R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$. **[Cauchy's Remainder]**

**Proof.** By the given hypotheses; $f, f', f'', \ldots, f^{n-1}$ are all continuous in $[a, a+h]$ and derivable in $]a, a+h[$. ...(1)

We define a function $\varphi$ on $]a, a+h[$ as follows :

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(a+h-x) \quad \forall\ x \in [a, a+h], \quad \ldots(2)$$

where $A$ is constant to be determined by $\phi(a) = \phi(a+h)$. ...(3)

From (2) and (3), we obtain

$$f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + Ah = f(a+h). \quad \ldots(4)$$

From (1) and (2), we see that

$\phi$ is continuous in $[a, a+h]$ and $\phi$ is derivable in $]a, a+h[$. Also $\phi(a) = \phi(a+h)$. Thus $\phi$ satisfies all the conditions of Rolle's Theorem and so there exists some $\theta$, $0 < \theta < 1$ such that

$$\phi'(a+\theta h) = 0. \quad \ldots(5)$$

**Differentiating (2) on both sides, we obtain**

$$\phi'(x) = f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - A$$

$$\therefore \quad 0 = \phi'(a+\theta h) = \frac{(h-\theta h)^{n-1}}{(n-1)!} f^n(a+\theta h) - A, \text{ using (5)}$$

or
$$A = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h).$$

Putting this value of $A$ in (4), we obtain

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h).$$

**Ex. 1.** *State and prove Taylor's Theorem with Lagrange's form of remainder.* (D.U., B.Sc. (G) 1995)

**Ex. 2.** *State and prove Taylor's Theorem with Cauchy's form of remainder.* (D.U., B.A. (P) 1999)

### 7.7 Maclaurin's Theorem

On taking $a = 0$ in Taylor Theorem, we get **Maclaurin's Theorem**, which may be stated as follows :

*If a function f defined on $[0, h]$ is such that*

*(i) $f^{n-1}$ is continuous in $[0, h]$,*

*(ii) $f^{n-1}$ is derivable in $]0, h[$,*

*then for each x in $[0, h]$, there exists a real number $\theta$ between 0 and 1 such that*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x).$$

*(Lagrange's remainder)*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x). \quad \textit{(Cauchy's remainder)}$$

### 7.8 Maclaurin's Infinite Series

Let $\quad S_n = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots \frac{x^{n-1}}{(n-1)!} f^{n-1}(0),$

$$R_n = \frac{x^n}{n!} f^n(\theta x) \text{ or } R_n = \frac{x^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x).$$

$\therefore \quad f(x) = S_n + R_n.$

Now $\quad R_n \to 0$ as $n \to \infty$ if and only if $f(x) = \lim\limits_{n \to \infty} S_n$.

Thus we have the following :

*If a function f defined on $[0, h]$ is such that* *(i)* $f^n$ *exists for each n i.e., f has derivatives of all orders,*

*(ii)* $R_n \to 0$ *as* $n \to \infty$, *then*

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\ldots+\frac{x^n}{n\,!}f^n(0)+\ldots$$

This is called **Maclaurin's infinite series expansion** of $f(x)$.

**Note.** *The conditions (i) and (ii) above must be satisfied for a func*
*to have Maclaurin's infinite series expansion.*

### 7.9. Power Series Expansions of Some Standard Functions

### 7.9.1. Obtain Maclaurin's series expansion of $e^x$.

[D.U., Maths (H) 1992 ; B.Sc. (G) 1998, 92, 9

Let $f(x)=e^x$. Then

$f'(x)=e^x, f''(x)=e^x, f'''(x)=e^x, \ldots, f^n(x)=e^x \ \forall \ n \in \mathbf{N}$.

Thus $f$ has derivatives of all orders.

Further $f'(0)=1, f''(0)=1, f'''(0)=1$ and so on.

The Lagrange's remainder after $n$ terms is

$$R_n=\frac{x^n}{n\,!}f^n(\theta x),\ 0<\theta<1$$

or

$$R_n=\frac{x^n}{n!}e^{\theta x},\ 0<\theta<1. \qquad (\because f^n(x)=e$$

We shall show that $R_n \to 0$ as $n \to \infty$.

Let $$A_n=\frac{x^n}{n\,!} \Rightarrow A_{n+1}=\frac{x^{n+1}}{(n+1)\,!}.$$

$$\therefore \quad \frac{A_{n+1}}{A_n}=\frac{x^{n+1}}{(n+1)\,!}\cdot\frac{n\,!}{x^n}=\frac{x}{n+1}$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{A_{n+1}}{A_n}\right)=0 \Rightarrow \lim_{n\to\infty} A_n=0 \Rightarrow \lim_{n\to\infty}\frac{x^n}{n\,!}=0.$$

[See Example 50, Chapter 3]

$$\therefore \quad \lim_{n\to\infty} R_n=\left(\lim_{n\to\infty}\frac{x^n}{n\,!}\right)e^{\theta x}=0.$$

Thus both the conditions of Maclaurin's series expansion are satisfied.

$$\therefore \quad f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots$$

Hence $$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots$$

**Ex.** ***Obtain Maclaurin's series expansion of*** $e^{2x}$.

### 7.9.2. Obtain Maclaurin's series expansion of sin $x$.

[D.U., B.Sc. (G) 1997, 93 ; Maths (H) 1993 ;
B.A. (P) 1998, 95, 92, 91]

Let $f(x)=\sin x \ \forall \ x \in \mathbf{R}$.

$$\therefore \quad f^n(x)=\sin\left(x+\frac{n\pi}{2}\right) \ \forall \ n \in \mathbf{N}.$$

Thus $f$ has derivatives of all orders.

Now $$|f^n(\theta x)|=\left|\sin\left(\theta x+\frac{n\pi}{2}\right)\right| \le 1 \ \forall \ n \text{ and } \forall \ x \in \mathbf{R}.$$

$$\therefore \quad |R_n|=\left|\frac{x^n}{n\,!}f^n(\theta x)\right| \le \left|\frac{x^n}{n\,!}\right| \to 0 \text{ as } n \to \infty.$$

Thus $R_n \to 0$ as $n \to \infty$ and so by Maclaurin's series,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\ldots\ldots$$

or $$\sin x=0+x\,.\,1+\frac{x^2}{2!}\,.\,0+\frac{x^3}{3!}(-1)\ldots\ldots$$

Hence $$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!}+\ldots\ldots \ \forall \ x \in \mathbf{R}.$$

**Ex.** *Obtain Maclaurin's series expansion of* $f(x)=\sin 2x$.

[D.U., B.A. (P) 1996]

### 7.9.3. Obtain Maclaurin's series expansion of cos $x$.

[D.U., B.Sc. (G) 1995, 90 ; Physics (H) 1994 ; B.A. (P) 1994]

Let $f(x)=\cos x$.

$$\therefore \quad f^n(x)=\cos\left(x+\frac{n\pi}{2}\right) \ \forall \ n \in \mathbf{N}.$$

Now $$|R_n|=\left|\frac{x^n}{n\,!}f^n(\theta x)\right|$$

$$=\left|\frac{x^n}{n\,!}\right|\left|\cos\left(\theta x+\frac{n\pi}{2}\right)\right|$$

$$\le \left|\frac{x^n}{n\,!}\right|.\,1 \to \text{ as } n \to \infty.$$

$\therefore R_n \to 0$ as $n \to \infty$. Applying Maclaurin's series,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\ldots\ldots$$

Now $\quad 0 < 1 - \theta < 1 + \theta x \Rightarrow 0 < \dfrac{1-\theta}{1+\theta x} < 1$

$$\Rightarrow \quad \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0 \text{ as } n \to \infty.$$

Also $\qquad x^n \to 0 \text{ as } n \to \infty \qquad (\because \; -1 < x < 0),$

and $\qquad \dfrac{1}{1+\theta x} < \dfrac{1}{1-|x|}$. Using these results in (2), we get

$$R_n \to 0 \text{ as } n \to \infty.$$

Thus $\quad R_n \to 0$ as $n \to \infty$ for $-1 < x \leq 1$.

By Maclaurin's series, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

We have $\quad f^n(0) = (-1)^{n-1}(n-1)!$ and $f(0) = 0$.

Hence $\quad \log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\ldots, \quad -1 < x \leq 1.$

**7.9.5 Obtain Maclaurin's series expansion of $(1+x)^m$, $-1 < x < 1$.**

[D.U., Maths (H) 1998, 92 ; B.Sc. (G) 1999 ; **B.C.A. 1998**]

Let $\qquad f(x) = (1+x)^m$.

**Case I.** *Let m be a positive integer.*

We have $\quad f'(x) = m(1+x)^{m-1}, f''(x) = m(m-1)(1+x)^{m-2}, \ldots\ldots$

$$f^m(x) = m(m-1)\ldots\ldots 3.2.1 = m! \text{ and } f^n(x) = 0 \text{ for } n > m.$$

Thus, by Maclaurin's expansion,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^m}{m!} f^m(0).$$

Hence $\quad (1+x)^m = 1 + mx + \dfrac{m(m-1)}{2!} x^2 + \ldots\ldots\ldots + x^m.$

**Case II.** *Let m be not a positive integer.*

Then $\quad f^n(x) = m(m-1)(m-2)\ldots(m-n+1)(1+x)^{m-n}, \quad \forall\, n \in \mathbf{N},$ and $x \neq -1$.

The Cauchy's form of remainder is

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^n f^n(\theta x), \; 0 < \theta < 1$$

$$= \frac{m(m-1)\ldots\ldots(m-n+1)}{(n-1)!} x^n (1-\theta)^{n-1} (1+\theta x)^{m-n},$$

$$\Rightarrow \quad 1-\frac{x^2}{2}\cos(\theta x) \geq 1-\frac{x^2}{2} \Rightarrow \cos x \geq 1-\frac{x^2}{2}, \text{ by (1).}$$

$$\therefore \quad 1-\frac{x^2}{2} \leq \cos x. \qquad \text{...(2)}$$

On applying Taylor's theorem to the function $f(x) = \cos x$ in $[0, x]$ with remainder after four terms, we get

$$f(x) = f(0) + f'(0) + \frac{x^2}{2\,!}f''(0) + \frac{x^3}{3\,!}f'''(0) + \frac{x^4}{4\,!}f^{iv}(\theta' x),\ 0 < \theta' < 1$$

or

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24}(\cos \theta' x),\ 0 < \theta' < 1. \qquad \text{...(3)}$$

Since $\cos(\theta' x) \leq 1$, for $0 < \theta' < 1$ and $x > 0$,

$$\therefore \quad 1 - \frac{x^2}{2} + \frac{x^4}{24}\cos(\theta' x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$\Rightarrow \quad \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}, \text{ using (3).}$$

Hence $1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$, $x > 0$.

**Case III.** Let $x < 0$. Then $-x > 0$ or $y > 0$, where $y = -x$.

By case II, $\quad 1 - \frac{y^2}{2} \leq \cos y \leq 1 - \frac{y^2}{2} + \frac{y^4}{24}$. $\qquad (\because\ y > 0)$

Putting $y = -x$, we obtain

$$1 - \frac{x^2}{2} \leq \cos(-x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Hence $1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$, $\forall\, x \in \mathbf{R}$.

**Example 60.** *Show that* $\sin x$ *lies between*

$$x - \frac{x^3}{6} \textit{ and } x - \frac{x^3}{6} + \frac{x^5}{120}. \qquad \text{[D.U., Maths (H) 1994]}$$

**Solution.** If $x = 0$, then there is nothing to prove because each of the expressions

$$x - \frac{x^3}{6},\ \sin x \quad \text{and} \quad x - \frac{x^3}{6} + \frac{x^5}{120}$$

has the value 0.

**Case I.** Let $x > 0$.

On applying Taylor's theorem to the function $f(x) = \sin x$ in $[0, x]$ with remainder after three terms, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2\,!}f''(0) + \frac{x^3}{3\,!}f'''(\theta x),\ 0 < \theta < 1$$

**Example 62.** *Using Taylor's theorem, or otherwise, prove that* :

$$1+x+\frac{x^2}{2}<e^x<1+x+\frac{x^2}{2}e^x,\ x>0. \qquad \text{[D.U., Physics (H) 1997]}$$

**Solution.** Applying Taylor's theorem to the function $f(x)=e^x$ in $[0, x]$ with remainder after two terms, we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(\theta x),\ 0<\theta<1$$

or

$$e^x=1+x+\frac{x^2}{2}e^{\theta x}. \qquad \text{...(1)}$$

Now $0<\theta<1$ and $x>0 \Rightarrow \theta x<x \Rightarrow e^{\theta x}<e^x$

$$\Rightarrow \quad 1+x+\frac{x^2}{2}e^{\theta x}<1+x+\frac{x^2}{2}e^x. \qquad \text{...(2)}$$

From (1) and (2), $e^x<1+x+\dfrac{x^2}{2}e^x$.

Again on applying Taylor's theorem to the function $f(x)=e^x$ in $[0, x]$ with remainder after three terms, we get

$$e^x=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}e^{\theta_1 x},\ 0<\theta_1<1$$

$$>1+x+\frac{x^2}{2},\ \text{as } x>0.$$

Hence $1+x+\dfrac{x^2}{2}<e^x<1+x+\dfrac{x^2}{2}e^x,\ x>0.$

**Example 63.** *Prove that* : $1+x<e^x<1+xe^x$, *if* $x>0$.
**Hint.** Similar to Example 62.
**Example 64.** *Show that* :

$$1-x<e^{-x}<1-x+\frac{x^2}{2},\ x>0.$$

**Hint.** Similar to Example 62. Take $f(x)=e^{-x}$. Apply Taylor's theorem to $f(x)$ in $[0, x]$ with remainder after two and three terms.

**Example 65.** *Prove that* :

$$0\le \sin x-\left(x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!}\right)\le\frac{x^9}{9\,!},\ \textit{for}\ x>0.$$

**Solution.** Refer to Example 60. By Taylor's Theorem,

$$\sin x=x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!}\cos(\theta_1 x),0<\theta_1<1.$$

Also

$$\sin x=x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!}+\frac{x^9}{9\,!}\cos(\theta_2 x),\ 0<\theta_2<1.$$

We know $\cos(\theta_1 x)\le 1$ and $\cos(\theta_2 x)\le 1\ \forall\ \theta_1 x$ and $\theta_2 x$. For $x>0$, Taking the limit as $x \to 1-0$ on both sides of (1), we get

$$0 = 1 - \frac{5}{2} + \frac{15}{8}(1-\theta)^{1/2} \text{ or } (1-\theta)^{1/2} = \frac{4}{5} \text{ or } 1-\theta = \frac{16}{25}.$$

Hence $\lim\limits_{x \to 1-0} \theta = \frac{9}{25} = 0.36.$

**Example 68.** *If* $f(x) = (1+x)^{1/3}, -1 < x < 1$, *show that*

$$(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{x^2}{9} + \frac{10}{81}(1+\theta x)^{-8/3}, 0 < \theta < 1.$$

**Please try yourself.**

**Example 69.** *If* $a > 0$, $h > 0$ *and* $n \in \mathbf{N}$, *prove that there exists* $\theta$, $0 < \theta < 1$, *such that*

$$\frac{1}{a+h} = \frac{1}{a} - \frac{h}{a^2} + \frac{h^2}{a^3} + \ldots + \frac{(-1)^{n-1} h^{n-1}}{a^n} + \frac{(-1)^n h^n}{(a+\theta h)^{n+1}}.$$

**Solution.** Let $f(x) = \frac{1}{x}, x > 0$. Then

$$f'(x) = -\frac{1}{x^2}, f''(x) = \frac{2}{x^3}, f'''(x) = \frac{-2.3}{x^4}, \ldots$$

$$f^n(x) = (-1)^n \frac{1.2.3 \ldots n}{x^{n+1}}.$$

By Taylor's theorem, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$
$$+ \frac{h^n}{n!}f^n(a+\theta h), 0 < \theta < 1.$$

Hence $\frac{1}{a+h} = \frac{1}{a} - \frac{h}{a^2} + \frac{h^2}{a^3} - \ldots + \frac{(-1)^{n-1} h^{n-1}}{(n-1)!} + \frac{(-1)^n h^n}{(a+\theta h)^{n+1}}.$

**Example 70.** *If* $f''$ *is continuous on* $[a-\delta, a+\delta[$ *for some* $\delta > 0$, *prove that*

$$\lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f(a).$$

**Solution.** By Taylor's theorem with remainder after two terms, for any $h$ such that $0 < h < \delta$, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2}f''(a+\theta_1 h), 0 < \theta_1 < 1$$

and $$f(a-h) = f(a) - hf'(a) + \frac{h^2}{2}f''(a-\theta_2 h), 0 < \theta_2 < 1.$$

From the above two relations, we obtain

$$f(a+h) - 2f(a) + f(a-h) = \frac{h^2}{2}[f''(a+\theta_1 h) + f''(a-\theta_2 h)].$$

**Example 72.** *Assuming the validity of expansion, show that*

$$\tan^{-1} x = \tan^{-1}\frac{\pi}{4} + \frac{\left(x-\frac{\pi}{4}\right)}{\left(1+\frac{\pi^2}{16}\right)} - \frac{\pi\left(x-\frac{\pi}{4}\right)^2}{4\left(1+\frac{\pi^2}{16}\right)^2} + \ldots\ldots$$

**Solution.** Let

$$f(x) = \tan^{-1} x = \tan^{-1}\left(\frac{\pi}{4} + x - \frac{\pi}{4}\right) = \tan^{-1}(a+h),$$

where $a = \pi/4,\ h = x - \pi/4$ and $a + h = x$.

Assuming the validity of expansion, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2\,!}f''(a) + \ldots..$$

$$\therefore\ \tan^{-1} x = \tan^{-1}\frac{\pi}{4} + \frac{(x-\pi/4)}{\{1+(\pi^2/16)\}} - \frac{\pi}{4}\,\frac{(x-\pi/4)^2}{\{1+(\pi^2/16)\}^2} + \ldots$$

$$\left[\because\ f'(x) = \frac{1}{1+x^2},\ f''(x) = -\frac{2x}{(1+x^2)^2}\ \text{etc.}\right]$$

**Example 73.** *Assuming the validity of expansion, show that*

$$\sin x = 1 - \frac{(x-\pi/2)^2}{2\,!} + \frac{(x-\pi/2)^4}{4\,!} - \ldots$$

**Solution.** Let $f(x) = \sin x = \sin\left(\frac{\pi}{2} + x - \frac{\pi}{2}\right) = \sin(a+h),$

where $a = \pi/2,\ h = x - \pi/2$ and $a + h = x$.

Assuming the validity of expansion, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3\,!}f'''(a) + \frac{h^4}{4\,!}f^{iv}(a) + \ldots \qquad \ldots(1)$$

Now $f(x) = \sin x \Rightarrow f'(x) = \cos x,\ f''(x) = -\sin x,$

$$f'''(x) = -\cos x,\ f^{iv}(x) = \sin x.$$

For $a = \pi/2$ ; $f(a) = 1,\ f'(a) = 0,\ f''(a) = -1,\ f'''(a) = 0,\ f^{iv}(a) = 1.$

Substituting these values in (1), we obtain

$$\sin x = 1 - \frac{(x-\pi/2)^2}{2\,!} + \frac{(x-\pi/2)^4}{4\,!} - \ldots..$$

**Example 74.** *Assuming the validity of expansion, show that*

$$\sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{\sqrt{2}}\left(1 + \theta - \frac{\theta^2}{2\,!} - \frac{\theta^3}{3\,!} + \ldots..\right).$$

**Solution.** Let $a = \pi/4, h = \theta, f(x) = \sin x$.

Assuming the validity of expansion, we have

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots\ldots$$

or $$\sin(a+h) = \sin a + h\cos a + \frac{h^2}{2!}(-\sin a) + \frac{h^3}{3!}(-\cos a) + \ldots\ldots$$

Putting $a = \pi/4$ and $h = \theta$, we obtain

$$\sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{\sqrt{2}} + \frac{\theta}{\sqrt{2}} + \frac{\theta^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\theta^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \ldots\ldots$$

Hence $$\sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{\sqrt{2}}\left(1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \ldots\ldots\right).$$

**Example 75.** *Assuming the possibility of expansion, show that*

(*a*) $\sin x = \frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) - \frac{(x-\pi/4)^2}{2!} - \frac{(x-\pi/4)^3}{3!} \ldots\ldots\right].$

**Hint.** Proceed like Example 73.

(*b*) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots$

(*c*) $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots$

**Hint.** Use $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$ in parts (*b*) and (*c*).

# 8

# Maxima and Minima

We recall that the *left hand* and *right hand derivatives* of a function at $x = c$ are defined as

$$Lf'(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad Rf'(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$$

respectively. Further $f'(c)$ exists iff $Lf'(c) = Rf'(c)$.

## 1 Maximum and Minimum Values of a Function

**Definition 1.** *A function f is said to have a* **maximum** *at a point c, if there exists some $\delta > 0$ such that*

$$f(x) < f(c) \quad \forall \; x \in \;]c - \delta, c + \delta[, x \neq c.$$

*If f has a maximum at c, we also say that f(c) is a* **maximum value** *of f.*

**Definition 2.** *A function f is said to have a* **minimum** *at a point c, if there exists some $\delta > 0$ such that*

$$f(x) > f(c) \quad \forall \; x \in \;]c - \delta, c + \delta[, x \neq c.$$

*If f has a minimum at c, we also say that f(c) is a* **minimum value** *of f.*

**Definition 3.** *A function f is said to have an* **extreme value** *at a point c iff f(c) is either a maximum value or a minimum value of f.*

**Theorem 1.** (*A necessary condition for an extreme value*)

*If a function f has an extreme value at a point c, then $f'(c) = 0$,* ***provided*** *f'(c)* ***exists.***

**Proof.** Suppose that $f$ has a maximum at $c$. Then there exists some $\delta > 0$ such that

$$f(x) < f(c) \quad \forall\, x \in \;]c - \delta, c + \delta[, x \neq c \qquad \ldots(1)$$

$$\Rightarrow \quad f(x) - f(c) < 0 \;\forall\, x \in \;]c - \delta, c + \delta[, x \neq c$$

$$\Rightarrow \quad \frac{f(x) - f(c)}{x - c} < 0, \text{ if } x - c > 0 \text{ and } x \in \;]c - \delta, c + \delta[$$

$$\Rightarrow \quad \frac{f(x) - f(c)}{x - c} < 0, \text{ if } c < x < c + \delta$$

$$\Rightarrow \quad \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \leq 0 \quad \Rightarrow \quad Rf'(c) \leq 0. \qquad \ldots(2)$$

Again by (1), $$\frac{f(x) - f(c)}{x - c} > 0, \text{ if } x - c < 0 \text{ and } x \in \;]c - \delta, c + \delta[$$

**Ex. 1.** *Justify, by giving an example, the following statements :*

(*i*) *A function may have an extreme value at a point without being derivable thereat.*

**Hint.** See Remark 2 above.

(*ii*) *The vanishing of the derivative at a point is not a sufficient condition for the point to be an extreme point.*

**Hint.** See Remark 1 above.

**Ex. 2.** *Give an example of a function f such that* $f'(c) = 0$ *at a point c but f has neither maxima nor minimum at* $x = c$.

[B.C.A. Sample Problem]

**Hint.** See Remark 1 above.

## 8.2 Sufficient Conditions for Extreme Values

**Theorem 3.** (*First derivative test for extreme values*)

*Let a function f be derivable in a neighbourhood of c, where f has an extreme value at c. Then f (c) is a maximum value if the sign of f′ changes from plus to minus and f (c) is a minimum value if the sign of f′ changes from minus to plus as x passes through c.*

**Proof.** Suppose $f'$ changes sign from plus to minus as $x$ passes through $c$. Then there exists a $\delta > 0$ such that $f'(x) > 0$ in $]c - \delta, c[$ and $f'(x) < 0$ in $]c, c + \delta[$. By Theorem 2, we see that $f$ is strictly increasing in $]c - \delta, c[$ and $f$ is strictly decreasing in $]c, c + \delta[$. Consequently,

$$f(x) < f(c) \quad \forall \quad x \in \,]c - \delta, c[, \qquad \text{...(1)}$$

and

$$f(x) < f(c) \quad \forall \quad x \in \,]c, c + \delta[. \qquad \text{...(2)}$$

(1) and (2) imply that $f(x) < f(c)\ \forall\ x \in \,]c - \delta, c + \delta[, x \neq c$.

Hence $f$ has a maximum value at $x = c$. Similarly, it can be proved that $f$ has a minimum value at $x = c$ if $f'$ changes sign from minus to plus as $x$ passes through $c$.

**Remark.** If $f'(x)$ does not change sign *i.e.*, $f'(x)$ keeps the same sign in a neighbourhood of $c$, say $]c - \delta, c + \delta[$, then $f(x)$ is either strictly increasing or strictly decreasing in $]c - \delta, c + \delta[$, which means that $f(c)$ is not an extreme value.

**Example 1.** *Examine the function* $x^3 - 6x^2 + 9x - 4$ *for maximum and minimum values.*

**Solution.** Let $f(x) = x^3 - 6x^2 + 9x - 4$. Then

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$$

or

$$f'(x) = 3(x - 1)(x - 3)$$

$\therefore$

$$f'(x) = 0 \text{ for } x = 1, 3.$$

Now for $x < 1, f'(x) = (-) \cdot (-) > 0$,

and for $x > 1, f'(x) = (+) \cdot (-) < 0$.

Thus $f'(x)$ changes sign from plus to minus in going through $x = 1$. Hence the function has a maximum at $x = 1$ and $f(1) = 0$ is a maximum value of $f$.

**Theorem 5.** *Let $f$ be a function defined on an interval $[a, b]$ and $a < c < b$. Let*

$$f'(c) = f''(c) = \ldots = f^{n-1}(c) = 0 \text{ and } f^{n}(c) \neq 0.$$

(*i*) *If $n$ is odd, then $f$ has neither a maximum nor a minimum at $x = c$.*

(*ii*) *If $n$ is even, then $f$ has a maximum at $x = c$ if $f^{n}(c) < 0$, and $f$ has a minimum at $x = c$ if $f^{n}(c) > 0$.*

## EXAMPLES

**Example 2.** *Find the maximum and minimum values of the function*

$$f(x) = 12x^5 - 45x^4 + 40x^3 + 6, x \in \mathbf{R}.$$

[D.U., B.Sc. (G) 1993]

**Solution.** We have

$$f'(x) = 60x^4 - 180x^3 + 120x^2$$
$$= 60x^2(x^2 - 3x + 2) = 60x^2(x - 1)(x - 2).$$

The extreme values of the function $f$ are given by $f'(x) = 0$

*i.e.*, $x = 0, 1, 2.$

Now $f''(x) = 240x^3 - 540x^2 + 240x$

or $f''(x) = 60(4x^3 - 9x^2 + 4x).$

$\therefore$ $f''(2) = 240 > 0 \Rightarrow f$ has a minimum at $x = 2.$

and $f''(1) = -60 < 0 \Rightarrow f$ has a maximum at $x = 1.$

Since $f''(0) = 0$, we find out the third derivative. We have

$$f'''(x) = 720x^2 - 1080x + 240 \Rightarrow f'''(0) = 240 \neq 0.$$

So $f$ has neither a maximum nor a minimum at $x = 0.$

Minimum value $= f(2) = -10.$

Maximum value $= f(1) = 12 - 45 + 40 + 6 = 13.$

**Example 3.** *Find the maximum and minimum values of the function*

$$f(x) = x^4 + 4x^3 - 2x^2 - 12x + 7.$$

[D.U., B.Sc. (G) 1992]

**Solution.** $f'(x) = 4x^3 + 12x^2 - 4x - 12 = 4(x^3 + 3x^2 - x - 3)$

$$= 4(x - 1)(x^2 + 4x + 3) = 4(x - 1)(x + 1)(x + 3).$$

The extreme values of $f$ are given by $f'(x) = 0.$

*i.e.*, $x = 1, -1, -3.$

Now $f''(x) = 4(3x^2 + 6x - 1), f'''(x) = 4(6x + 6).$

$f''(1) = 32 > 0 \Rightarrow f$ has a minimum at $x = 1,$

$f''(-1) = -16 < 0 \Rightarrow f$ has a maximum at $x = -1,$

$f''(-3) = 32 > 0 \Rightarrow f$ has a minimum at $x = -3.$

Thus $f$ has a minimum at $x = 1$ and $-3$, and a maximum at $x = -1.$

Minimum value $= f(1) = f(-3) = -2.$

Maximum value $= f(-1) = 14.$

**Example 4.** *Show that the function f defined by*

$$f(x)=x^5-5x^4+5x^3-1 \;\forall\; x \in \mathbf{R}$$

*has a maximum value when $x=1$, a minimum value when $x=3$ and neither when $x=0$.* [D.U., B.Sc. (G) 1991]

**Solution.** $f'(x)=5x^4-20x^3+15x^2$

$$=5x^2(x^2-4x+3)=5x^2(x-1)(x-3).$$

The extreme values of $f$ are given by $f'(x)=0$

*i.e.,* $x=0, 1, 3.$

Now $f''(x)=20x^3-60x^2+30x, f'''(x)=60x^2-120x+30.$

$f''(1)=-10<0 \Rightarrow f$ has a maximum at $x=1$,

$f''(3)=90>0 \Rightarrow f$ has a minimum at $x=3$.

Since $f''(0)=0$ and $f'''(0)=30 \neq 0$, $f$ has neither a maximum nor a minimum at $x=0$.

**Example 5.** *Given n real numbers $a_1, a_2, \ldots, a_n$, find the value of x for which the sum $\sum_{i=1}^{n}(x-a_i)^2$ is a minimum.* [D.U., B.Sc. (G) 1990]

**Solution.** Let $f(x)=\sum_{i=1}^{n}(x-a_i)^2$.

$$\therefore \quad f'(x)=\sum_{i=1}^{n}2(x-a_i)=2nx-2(a_1+a_2+\ldots+a_n).$$

The arithmetic mean $(\bar{a})$ of $a_1, a_2, \ldots, a_n$ is given by

$$\bar{a}=\frac{1}{n}(a_1+a_2+\ldots+a_n) \quad i.e., \quad n\bar{a}=(a_1+a_2+\ldots+a_n).$$

$\therefore \; f'(x)=2nx-2n\bar{a}$ and $f''(x)=2n>0.$

The extreme values of $f$ are given by $f'(x)=0$

$\Rightarrow \quad 2nx-2n\bar{a}=0 \Rightarrow x=\bar{a}.$

Since $f''(x)=2n>0$, $f$ has a minimum at $x=\bar{a}$.

**Example 6.** *Show that the function $f(x)=\sin x\,(1+\cos x)$ has a maximum value when $x=\frac{1}{3}\pi$.*

[D.U., Maths (H) 1996 ; B.C.A. 1998 ; B.Sc. (G) 1997]

**Solution.** $f'(x)=\cos x(1+\cos x)-\sin^2 x$

$$=\cos x+\cos^2 x-(1-\cos^2 x)$$

$$=2\cos^2 x+\cos x-1.$$

$\therefore \quad f'(x)=0 \quad \Rightarrow 2\cos^2 x+\cos x-1=0$

$\Rightarrow (2\cos x-1)(\cos x+1)=0$

$\Rightarrow \cos x=\frac{1}{2}, \cos x=-1$

$\Rightarrow x=\frac{1}{3}\pi, \pi.$

Now $f''(x)=-4\cos x\sin x-\sin x=-\sin x(4\cos x+1)$

$$\therefore \quad f''(\tfrac{1}{3}\pi)=-\frac{\sqrt{3}}{2}(4\cdot\tfrac{1}{2}+1)=-\frac{3}{2}\sqrt{3}<0.$$

Hence $f$ has a maximum at $x=\frac{1}{3}\pi$.

**Note.** It can be verified that $f''(\pi)=0$ and $f'''(\pi)\neq 0$. So $f$ has neither a maximum nor a minimum at $x=\pi$.

**Example 7.** *Find the maxima and minima of the function $f(x)=\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x$, for all $x \in [0, \pi]$.*

[D.U., Maths (H) 1999, 92]

**Solution.** $f'(x)=\cos x+\cos 2x+\cos 3x$

$$=\cos 2x+2\cos 2x\cos x$$

$$=\cos 2x(1+2\cos x), x \in [0, \pi].$$

$\therefore \quad f'(x)=0 \Rightarrow \cos 2x=0 \Rightarrow 2x=\frac{1}{2}\pi, \frac{3}{2}\pi \Rightarrow x=\frac{1}{4}\pi, \frac{3}{4}\pi;$

and $\quad 1+2\cos x=0 \Rightarrow \cos x=-\frac{1}{2} \Rightarrow x=\frac{2}{3}\pi.$

Now $\quad f''(x)=-\sin x-2\sin 2x-3\sin 3x.$

$\therefore \quad f''(\frac{1}{4}\pi)=-\frac{1}{\sqrt{2}}-2-\frac{3}{\sqrt{2}}<0 \Rightarrow f$ has a maximum at $x=\frac{1}{4}\pi$.

$$f''(\tfrac{3}{4}\pi)=-\frac{1}{\sqrt{2}}+2-\frac{3}{\sqrt{2}}=2-2\sqrt{2}<0$$

and so $f$ has a maximum at $x=\frac{3}{4}\pi$.

$$f''(\tfrac{2}{3}\pi)=-\frac{\sqrt{3}}{2}+2\cdot\frac{\sqrt{3}}{2}-0=\frac{\sqrt{3}}{2}>0.$$

Thus $f$ has a minimum at $x=\frac{2}{3}\pi$.

$$\text{Maximum value}=f(\tfrac{1}{4}\pi)=\frac{1}{\sqrt{2}}+\frac{1}{2}\cdot 1+\frac{1}{3}\cdot\frac{1}{\sqrt{2}}$$

$$=\frac{4}{3\sqrt{2}}+\frac{1}{2}=\frac{4\sqrt{2}+3}{6}.$$

Now $$f(\tfrac{3}{4}\pi)=\frac{4\sqrt{2}-3}{6}.$$

$$\text{Minimum value}=f(\tfrac{2}{3}\pi)=\frac{\sqrt{3}}{2}-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}+0=\frac{\sqrt{3}}{4}.$$

**Example 8.** *Find the maximum and minimum values of the function $f(x)=4x^{-1}-(x-1)^{-1}$ for all $x \in \mathbf{R}-\{0, 1\}$.* [D.U., B.Sc. (G) 1995]

**Solution.** $f'(x)=-4x^{-2}+(x-1)^{-2}, f''(x)=8x^{-3}-2(x-1)^{-3}.$

The extreme values of $f$ are given by $f'(x)=0$

**Example 11.** *Obtain the points of maxima and minima for the function :*

$$f(x) = (x-3)^2 (x+3), \text{ for all } x \in \mathbf{R}$$

[D.U., B.A. (P) 1997, 95]

**Solution.** $f'(x) = (x-3)^2 + 2(x^2-9) = 3(x^2-2x-3) = 3(x+1)(x-3)$

$\therefore \quad f'(x) = 0 \Rightarrow x = -1, 3.$

Now $f''(x) = 6(x-1)$ and so $f''(3) = 12 > 0$ and $f''(-1) = -12 < 0$.

Hence $f(x)$ has a minimum at $x = 3$ and maximum at $x = -1$. Minimum value of $f(x) = 0$ (at $x = 3$) and maximum value of $f(x) = 32$ (at $x = -1$)

**Example 12.** *Prove that* $f(x) = \sin(x - \frac{1}{3}\pi) \sin x \sin(x + \frac{1}{3}\pi)$, *for all* $x \in \mathbf{R}$ *has a minimum at* $x = \pi/6$ *and maxima at* $x = \pi/2$, $x = -\pi/6$

**Solution.** We have

$$f(x) = \frac{1}{2}\left(\cos\frac{2\pi}{3} - \cos 2x\right)\sin x = -\left(\frac{1}{4} + \frac{1}{2}\cos 2x\right)\sin x$$

$$f'(x) = -\left[-\sin 2x \sin x + \left(\frac{1}{4} + \frac{1}{2}\cos 2x\right)\cos x\right]$$

$$= 2\sin^2 x \cos x - \frac{1}{4}\cos x - \frac{1}{2}(1 - 2\sin^2 x)\cos x$$

$$= 3\sin^2 x \cos x - \frac{3}{4}\cos x = 3\cos x\left(\sin^2 x - \frac{1}{4}\right)$$

$\therefore \quad f'(x) = 0 \Rightarrow \cos x = 0, \sin x = \pm\frac{1}{2} \Rightarrow x = \frac{\pi}{2}, \pm\frac{\pi}{6}.$

Now $\quad f''(x) = 6\sin x\cos^2 x - 3\sin^3 x + \frac{3}{4}\sin x.$

$f''(\pi/2) = 0 - 3 + \frac{3}{4} = -\frac{9}{4} < 0$. Thus $f(x)$ has a maximum at $x = \pi/2$.

$f''(\pi/6) = 6 \cdot \frac{1}{2} \cdot \frac{3}{4} - \frac{3}{8} + \frac{3}{8} = \frac{9}{4} > 0.$

Thus $f(x)$ has a minimum at $x = \pi/6$.

Finally, $f''(-\pi/6) = 6(-1/2)(3/4) + 3/8 - 3/8 = -9/4 < 0$.

Thus $f(x)$ has a maximum at $x = -\pi/6$.

**Example 13.** *Show that the function f, defined by*

$$f(x) = x^4\left(x - \frac{1}{2}\pi\right)^2 + \sin^4 x, \text{ for all } x \in \mathbf{R},$$

*has extreme values at* $x = 0$ *and* $x = \pi/2$ *and determine whether they are maxima and minima.*

**Solution.** We have

$$f'(x) = 4x^3\left(x - \frac{1}{2}\pi\right)^2 + 2x^4\left(x - \frac{1}{2}\pi\right) + 4\sin^3 x\cos x,$$

$$f''(x) = 12x^2\left(x - \frac{1}{2}\pi\right)^2 + 16x^3\left(x - \frac{1}{2}\pi\right) + 2x^4 - 4\sin^4 x + 12\sin^2 x\cos^2 x.$$

$$|x-3| = (x-3) \text{ if } x \geq 3, \; |x-3| = 3-x \text{ if } x < 3.$$

$$\therefore \qquad f(x) \begin{cases} 2(2-x)+5(3-x) = 19-7x, \text{ if } x < 2 \\ 5, \quad \text{if } x = 2 \\ 2(x-2)+5(3-x) = 11-3x, \text{ if } 2 < x < 3 \\ 2, \quad \text{if } x = 3 \\ 2(x-2)+5(x-3) = 7x-19, \text{ if } x > 3. \end{cases}$$

The graph of this function is given below :

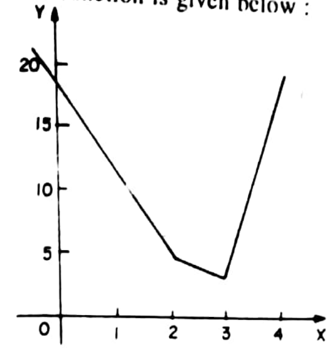

Fig. 3.

From the above graph, it is clear that the minimum value of the given function occurs at $x = 3$ only and this minimum value is 2. It can be verified that $f$ is not derivable at $x = 3$, but $f$ has the minimum at $x = 3$.

**Example 16.** *Prove that the function f defined by*

$$f(x) = 3|x| + 4|x-1| \; \forall \, x \in \mathbf{R}$$

*has a minimum value, and that this value is* 3. [D.U., B.Sc. (G) 1996]

**Solution.** We have

$$|x| = x \text{ if } x \geq 0, \; |x| = -x \text{ if } x < 0.$$

$$|x-1| = x-1 \text{ if } x \geq 1, \; |x-1| = 1-x \text{ if } x < 1.$$

$$\therefore f(x) = \begin{cases} 3(-x)+4(1-x) = 4-7x, & \text{if } x < 0 \\ 4 & \text{, if } x = 0 \\ 3(x)+4(1-x) = 4-x & \text{, if } 0 < x < 1 \\ 3 & \text{, if } x = 1 \\ 3(x)+4(x-1) = 7x-4 & \text{, if } x > 1. \end{cases}$$

The graph of this function is given below :

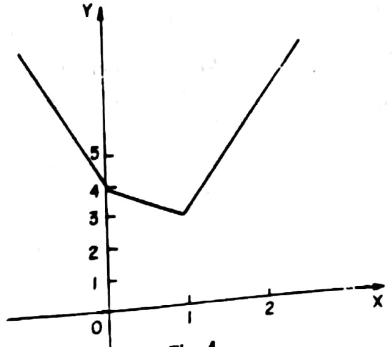

Fig. 4.

Thus $f'$ changes sign from negative to positive as $x$ passes through $b$ and so by Theorem 3, $f$ has a minimum at $x = b$.

Since $2n$ is an even integer, $(x - a)^{2n}$ does not change sign as $x$ passes through $a$ *i.e.*, $f'(x)$ does not change sign as $x$ passes through $a$. Hence $f$ has neither a maximum nor a minimum at $x = a$.

**Example 19.** *Obtain the extreme values of the function*

$$f(x) = x^3 + 3px + q \ \forall \ x \in \mathbf{R} \ ;$$

*p, q being fixed real numbers.*

**Solution.** $f'(x) = 3x^2 + 3p, f''(x) = 6x, f'''(x) = 6.$

The extreme values of $f(x)$ are given by $f'(x) = 0$

$$\Rightarrow \quad 3x^2 + 3p = 0 \Rightarrow x^2 + p = 0.$$

The equation $x^2 + p = 0$ has real roots iff $p \leq 0$.

Thus $f(x)$ has no extreme values if $p \geq 0$.

$$(p = 0 \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad f'(0) = f''(0) = 0 \quad \text{and} \quad f'''(0) \neq 0).$$

If $p < 0$, the extreme values of $f(x)$ are $\pm \sqrt{-p}$.

Since $f''(\sqrt{-p}) > 0$, $x = \sqrt{-p}$ gives a minimum value.

Since $f''(-\sqrt{-p}) < 0$, $x = -\sqrt{-p}$ gives a maximum value.

**Example 20.** *Show that the function f, defined by*

$$f(x) = x^p (1 - x)^q \text{ for all } x \in \mathbf{R},$$

*where p and q are positive integers, has a maximum value, whatever the values of p and q may be.* [D.U., Maths (H) 1994]

**Solution.** $f'(x) = px^{p-1}(1 - x)^q - qx^p(1 - x)^{q-1}$

$$= x^{p-1}(1 - x)^{q-1}[p(1 - x) - qx]$$

$$\therefore \quad f'(x) = x^{p-1}(1 - x)^{q-1}[p - x(p + q)], \quad \ldots(1)$$

$$f''(x) = (p - 1)x^{p-2}(1 - x)^{q-1}[p - x(p + q)] - (p + q)x^{p-1}(1 - x)^{q-1}$$
$$- (q - 1)x^{p-1}(1 - x)^{q-2}[p - x(p + q)]. \quad \ldots(2)$$

The extreme values of $f$ are given by $f'(x) = 0$

*i.e.,* $\quad x = 0, 1, p/(p + q)$, using (1).

$$\text{From (2),} \quad f''\left(\frac{p}{p + q}\right) = -(p + q)\left(\frac{p}{p + q}\right)^{p-1}\left(\frac{q}{p + q}\right)^{q-1} < 0,$$

since $p$ and $q$ are positive integers.

Hence the function $f$ has a maximum value at $x = p/(p + q)$, whatever the values of $p$ and $q$ may be. The maximum value of $f$ given by putting $x = p/(p + q)$ in $f(x) = x^p(1 - x)^q$ is $p^p q^q/(p + q)^{p+q}$.

**Example 21.** *Show that the function f, defined by*

$$f(x) = |x|^p |x - 1|^q, \text{ for all } x \in \mathbf{R},$$

*has a maximum value* $p^p q^q/(p + q)^{p+q}$, *p and q being positive integers.*

[D.U., B.Sc. (G) 1999 ; Maths (H) 1998]

**Solution.** We have

$$f(x) = \begin{cases} (-1)^p x^p (1-x)^q, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ x^p (1-x)^q, & \text{if } 0 < x < 1, \\ 0, & \text{if } x = 1, \\ x^p (x-1)^q, & \text{if } x > 1. \end{cases}$$

Consider $f(x) = x^p (1-x)^q, 0 < x < 1.$

By Example 20, $f$ has a maximum value at $p/(p+q)$.

The maximum value of $f$ is $p^p q^q/(p+q)^{p+q}$.

**Example 22.** *Show that the height of closed cylinder of given volume and least surface is equal to its diameter.* [D.U., Maths (H) 1995]

**Solution.** Let $h$ be the height and $r$ the radius of the circular base of the given closed cylinder. Its volume ($V$) and surface ($S$) are given by

$$V = \pi r^2 h,$$

$$S = 2\pi r h + 2\pi r^2$$

or
$$S = 2\pi r\left(\frac{V}{\pi r^2}\right) + 2\pi r^2 = \frac{2V}{r} + 2\pi r^2.$$

$$\therefore \quad \frac{dS}{dr} = -\frac{2V}{r^2} + 4\pi r, \quad \frac{d^2S}{dr^2} = \frac{4V}{r^3} + 4\pi.$$

Now
$$\frac{dS}{dr} = 0 \Rightarrow V = 2\pi r^3 \Rightarrow \pi r^2 h = 2\pi r^3 \Rightarrow h = 2r.$$

For
$$V = 2\pi r^3, \; d^2S/dr^2 = 12\,\pi > 0.$$

Hence the surface of the closed cylinder is least when $h = 2r$ *i.e.*, the height of the cylinder is equal to its diameter.

**Example 23.** *Given the perimeter of a rectangle, show that its area is maximum when it is a square.* [D.U., B.Sc. (G) 1998]

**Solution.** Consider a rectangle $ABCD$ of length $x$ and width $y$. Its perimeter ($P$) and area ($A$) are given by

$$P = 2(x+y), \; A = xy$$

or
$$A = x\left(\frac{P}{2} - x\right), \text{ where } P \text{ is fixed.}$$

$$\therefore \quad \frac{dA}{dx} = \frac{P}{2} - 2x \text{ and } \frac{d^2A}{dx^2} = -2 < 0.$$

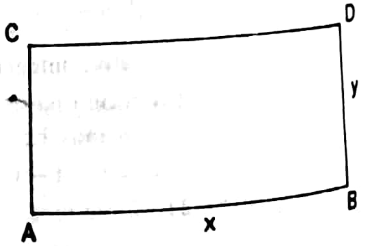

Fig. 6.

Hence $A$ is maximum when $dA/dx = 0$ *i.e.*, $P = 4x$ *i.e.*, $2x + 2y = 4x$ *i.e.*, $x = y$. Hence the area of the rectangle is maximum when it is a square.

**Example 24.** *Show that the semi-vertical angle of a cone of maximum volume and of given slant height is* $\tan^{-1}\sqrt{2}$.

**Solution.** Let $\theta$ be the semi-vertical angle and $l$ the slant height of the given cone. The volume of the cone of height $h$ and radius $r$ is $V = \frac{1}{3}\pi r^2 h$, where $r = l \sin\theta$ and $h = l\cos\theta$.

$$\therefore \quad V = \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta, \; l \text{ being fixed}$$

$$\frac{dV}{d\theta} = \frac{1}{3}\pi l^3 (2\sin\theta\cos^2\theta - \sin^3\theta)$$

$$= \frac{1}{3}\pi l^3 \sin\theta\,(2\cos^2\theta - \sin^2\theta)$$

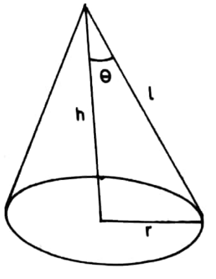

Fig. 7.

Now $dV/d\theta = 0 \Rightarrow 2\cos^2\theta - \sin^2\theta = 0$, since $\sin\theta \neq 0$, for $0 < \theta < \pi/2$

$$\Rightarrow \quad \tan\theta = \sqrt{2} \Rightarrow \theta = \tan^{-1}\sqrt{2}.$$

It can be verified that $d^2V/d\theta^2 < 0$, when $\tan\theta = \sqrt{2}$.

Hence the volume of the cone is maximum when $\theta = \tan^{-1}\sqrt{2}$.

**Example 25.** *Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius* $a$ *is* $2a/\sqrt{3}$.

[B.C.A. Sample Problem]

**Solution.** Let $O$ be the centre of the sphere of radius $a$. We inscribe in this sphere a cylinder of height $BC$ and radius $OA$. Let $\angle AOB = \theta$. Then

$$OA = a\cos\theta, \qquad AB = a\sin\theta,$$

$$BC = 2a\sin\theta.$$

The volume of the cylinder is

$$V = \pi\,(OA)^2\,(BC)$$

or
$$V = \pi a^2\cos^2\theta \,.\, 2a\sin\theta$$

or
$$V = 2\pi a^3\cos^2\theta\sin\theta.$$

$$\therefore \quad dV/d\theta = 2\pi a^3(-2\cos\theta\sin^2\theta + \cos^3\theta).$$

$$= 2\pi a^3\cos\theta\,(\cos^2\theta - 2\sin^2\theta).$$

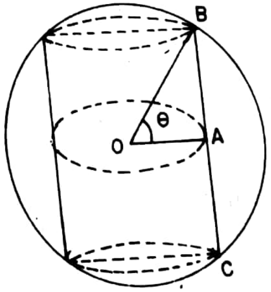

Fig. 8.

Now $dV/d\theta = 0 \Rightarrow \cos^2\theta - 2\sin^2\theta = 0$, since $\cos\theta \neq 0$ in $0 < \theta < \pi/2$

$$\Rightarrow \quad \tan\theta = 1/\sqrt{2} \Rightarrow \sin\theta = 1/\sqrt{3}.$$

It can be verified that $d^2V/d\theta^2 < 0$ for $\tan\theta = 1/\sqrt{2}$.

Hence the height of the cylinder of maximum volume is

$$BC = 2a\sin\theta = 2a/\sqrt{3}.$$

## EXERCISES

1. Show that $x^4 - 4x^3 + 6x^2 - 4x + 1$ has a minimum at $x = 1$.
2. Find the maximum and minimum values of the function $f(x) = (x-1)(x-2)(x-3)$ for all $x \in \mathbf{R}$.

   (**Ans.** Max. value $= 2/3\sqrt{3}$, Min. value $= -2/3\sqrt{3}$.)
3. Find the maximum and minimum values of $(1-x)^2 e^x$.

   (**Ans.** Max. value $= 4/e$, Min. value $= 0$.)
4. Show that $x^x$ is minimum for $x = e^{-1}$.

   [**Hint.** $y = x^x \Rightarrow \log y = x \log x \Rightarrow dy/dx = y(1 + \log x)$.

   $\therefore\ dy/dx = 0 \Rightarrow 1 + \log x = 0 \Rightarrow \log x = -1 = -\log e = \log e^{-1}$

   $\Rightarrow x = e^{-1}$.]
5. Obtain the maximum and minimum values of the function

   $$f(x) = 2x^3 - 15x^2 + 36x + 10 \text{ for } x \in \mathbf{R}.$$

   (**Ans.** Max. value = 38, Min. value = 37.)
6. Show that $x^5 - 5x^4 + 5x^3 - 10$ has a maximum at 1, a minimum at 3 and neither at 0. Also find maximum and minimum values. [D.U., B.Sc. (G) 1994]

   [**Hint.** See Example 4.]

   ( **Ans.** Max. value $= f(1) = -9$, Min. Value $= f(3) = -37$.)
7. Find the maxima and minima of the function :

   $f(x) = 4x^5 - 25x^4 + 40x^3 + 4$. [D.U., B.A. (P) 1996]

   (**Ans.** Max. value = 52 at $x = 2$, min. value = 31 at $x = 3$, neither maximum nor minimum at $x = 0$)
8. Find the maximum and the minimum as well as the greatest and the least values of the function $f$ where $f(x) = x^3 - 12x^2 + 45x$ in the interval $[0, 7]$. [D.U., B.A. (P) 1991]

   [**Hint.** $f'(x) = 0 \Rightarrow 3x^2 - 24x + 45 = 0 \Rightarrow x = 3, 5$.

   Now $f''(3) < 0$ and $f(3) = 54$, $f''(5) > 0$ and $f(5) = 50$.

   Also $f(0) = 0$, $f(7) = 70$. Greatest value = 70 and least value = 0. $f$ has minimum at $x = 5$ and maximum at $x = 3$].
9. Find the maxima and minima of the function

   $$f(x) = \cos x + \tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos 3x \text{ in } [0, \pi].$$

   (**Ans.** Max. at 0, $2\pi/3$ and Min. at $\pi/2$, $\pi$)

   [**Hint.** $f'(x) = -(\sin x + \sin 2x + \sin 3x) = -\sin 2x(1 + 2\cos x)$

   $f'(x) = 0 \Rightarrow 2x = 0, \pi, 2\pi \Rightarrow x = 0, \pi/2, \pi$;

   and $1 + 2\cos x = 0 \Rightarrow \cos x = -\tfrac{1}{2} \Rightarrow x = 2\pi/3$]
10. Investigate the maximum and minimum values of the function $f(x) = \sin x + \cos 2x$, $x \in \mathbf{R}$. [D.U., B.Sc. (G) 1989]

[**Hint.** $f'(x) = \cos x - 2\sin 2x = \cos x - 4\sin x\cos x = \cos x(1 - 4\sin x)$.

Now $\cos x = 0 \Rightarrow x = \pi/2, 3\pi/2$ and

$1 - 4\sin x = 0 \Rightarrow x = \sin^{-1}(\tfrac{1}{4}), \pi - \sin^{-1}(\tfrac{1}{4})$.

$f$ is minimum for $x = \pi/2, 3\pi/2$ and maximum for the other two values.]

11. Divide 100 into two parts such the sum of their squares is minimum.

    (**Ans.** 50, 50) [D.U., B.Sc. (G) 1987]

    [**Hint.** $y = x^2 + (100-x)^2 \Rightarrow dy/dx = 4x - 200$, $d^2y/dx^2 = 4 > 0$.]
12. Divide 24 into two parts such the product of one by the square of the other is maximum. (**Ans.** 8, 16) [D.U., B.Sc. (G) 1986]

    [Hint. $y = (24-x)x^2 \Rightarrow dy/dx = 84x - 3x^2$]
13. Divide 200 into two parts such that the product of one part with the cube of the second is maximum. (**Ans.** 50, 150) [B.C.A. Sample Problem]
14. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2\alpha$.

    [Hint. If $x$ is the radius of the base of the cylinder, then the height of the cylinder is $h - x\cot\alpha$. Volume of the cylinder is

    $$V = \pi x^2(h - x\cot\alpha). \text{ Find } dV/dx.]$$
15. Show that of all rectangles of given area, the square has the smallest perimeter.
16. Show that the rectangle of maximum perimeter which can be inscribed in a circle in a square.
17. Show that the rectangle of maximum area that can be inscribed in a circle is a square.
18. Show that the right circular cylinder of the given surface and maximum volume is such that its height is equal to the diameter of the base.

    [Hint. $V = \pi r^2 h$, $S = 2\pi r^2 + 2\pi rh$

    $$\therefore\quad V = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right). \text{ Find } \frac{dV}{dr}.]$$
19. Show that the curved surface of a right circular cylinder of greatest curved surface which can be inscribed in a given sphere is one half that of the sphere.

    [Hint. See Example 25.

    $S = 2\pi r h = 2\pi . OA . BC = 2\pi . a\cos\theta . 2a\sin\theta$ Find $dS/d\theta$.]
20. Prove that a conical tent of a given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.

    [Hint. If α is the semi-vertical angle and $r$ the radius of the base of the given conical tent, then

    $V = \frac{1}{3}\pi r^3\cot\alpha$, $S = \pi r^2\operatorname{cosec}\alpha$. Find $dS/d\alpha$, where V is fixed i.e., $dV/d\alpha = 0$.]

# 9

# Indeterminate Forms

## 9.1 Indeterminate Forms

We know that $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$,

provided $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist and $\lim\limits_{x\to a} g(x) \neq 0$.

However, quite often, it is possible to evaluate

$$\lim_{x\to a} \frac{f(x)}{g(x)}, \text{ even when } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0.$$

In this case $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is said to be the **indeterminate form** $\dfrac{0}{0}$.

Such a limit can be evaluated by **L'Hospital's Rule** as proved below:

Other indeterminate forms are

$$\frac{\infty}{\infty}, 0.\infty, \infty - \infty, 0^0, 1^\infty \text{ and } \infty^0.$$

## 9.2 L'Hospital's Rule for $\frac{0}{0}$ Form

*If f and g be two functions such that*

(*i*) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$,

(*ii*) $f'(x), g'(x)$ *exist and* $g(x) \neq 0,\ g'(x) \neq 0\ \forall\ x \in\ ]a-\delta, a+\delta[$, $\delta > 0$, *except possibly at a,*

(*iii*) $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ *exists, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

**Proof.** Let us take $f(a) = g(a) = 0$. ...(*iv*)

From (*i*) and (*iv*), we obtain

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a).$$

Thus $f$ and $g$ are continuous at the point $a$.

Consider the interval $[a, x]$, where $x > a$. We notice that

(*i*) $f$ and $g$ are continuous in $[a, x]$

(*ii*) $f$ and $g$ are derivable in $]a, x[$

(*iii*) $g'(t) \neq 0\ \forall\ t \in\ ]a, x[$.

Hence by Cauchy's mean value theorem, there exists some $c \in\ ]a, x[$ such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}$$

or

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}, c \in\ ]a, x[.$$

Since $c \to a$ as $x \to a$,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{c\to a} \frac{f'(c)}{g'(c)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

**Remark.** It may happen that $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ is $\dfrac{0}{0}$ form. In this case, we can extend the L'Hospital's rule as follows :

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)},$$

where $f'$ and $g'$ satisfy the conditions of L'Hospital's rule.

This rule can be generalized as follows :

*If f(x) and g(x) are functions such that* $\lim\limits_{x\to a} \dfrac{f^{(n-1)}(x)}{g^{(n-1)}(x)}$ *represents the indeterminate form* $\dfrac{0}{0}$ *and the functions* $f^{(n)}(x)$ *and* $g^{(n)}(x)$ *satisfy the conditions of L'Hospital's rule, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \ldots = \lim_{x\to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

## 9.3. L'Hospital's Rule for $\frac{\infty}{\infty}$ Form

*If f and g be two functions such that*

(*i*) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$,

(*ii*) $f'(x), g'(x)$ *exist and* $g(x) \neq 0,\ g'(x) \neq 0\ \forall\ x \in\ ]a-\delta, a+\delta[$, $\delta > 0$, *except possibly at a,*

(*iii*) $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ *exists, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

**Proof.** Take two points $x$ and $b$ satisfying $a < x < b < a + \delta$.

It is easily verified that $f$ and $g$ satisfy all the conditions of Cauchy's mean value theorem in $[x, b]$. Thus there exists a point $c \in ]x, b[$ such that

$$\frac{f(b)-f(x)}{g(b)-g(x)} = \frac{f'(c)}{g'(c)}$$

or

$$\frac{f(x)\left[\dfrac{f(b)}{f(x)} - 1\right]}{g(x)\left[\dfrac{g(b)}{g(x)} - 1\right]} = \frac{f'(c)}{g'(c)}. \quad \ldots(1)$$

Since $\lim\limits_{x \to a} f(x) = \infty$, $\lim\limits_{x \to a} \dfrac{f(b)}{f(x)} = 0$. Similarly, $\lim\limits_{x \to a} \dfrac{g(b)}{g(x)} = 0$.

Using in (1), we obtain

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}. \quad \left(\begin{array}{c} \because\ c \to a \\ \text{as } x \to a \end{array}\right)$$

**Remark.** The L'Hospital's rule for $\dfrac{\infty}{\infty}$ form can be extended to higher derivatives as done in the case of $\dfrac{0}{0}$ form.

## EXAMPLES

**Example 1.** *Evaluate* $\lim\limits_{x \to 0} \dfrac{xe^x - \log(1+x)}{x^2}$. [D.U., B.A. (P) 1989]

**Solution.** We have $\lim\limits_{x \to 0} [xe^x - \log(1+x)] = 0 = \lim\limits_{x \to 0} x^2$.

By L' Hospital's rule, we have

$$\lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2} = \lim_{x \to 0} \frac{(xe^x + e^x) - \frac{1}{1+x}}{2x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{xe^x + 2e^x + \frac{1}{(1+x)^2}}{2} = \frac{3}{2}.$$

**Example 2.** *Evaluate* $\lim\limits_{x \to 0} \dfrac{1 - \cos x^2}{x^2 \sin x^2}$.

**Solution.** We have $\dfrac{1-\cos x^2}{x^2 \sin x^2} = \dfrac{1 - \cos x^2}{x^4} \cdot \dfrac{x^2}{\sin x^2}$.

Since $\lim\limits_{x \to 0} \dfrac{x^2}{\sin x^2} = 1$, $\therefore$ $\lim\limits_{x \to 0} \dfrac{1-\cos x^2}{x^2 \sin x^2} = \lim\limits_{x \to 0} \dfrac{1-\cos x^2}{x^4} \left(\dfrac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{2x \sin x^2}{4x^3} \quad \text{(by L'Hospital's rule)}$$

$$= \lim_{x \to 0} \frac{1}{2} \cdot \frac{\sin x^2}{x^2} = \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

**Example 3.** *Evaluate* $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^2 \tan x}$.

[D.U., Maths (H) 1993 ; B.A. (P) 1998]

**Solution.** We have

$$\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x} = \lim_{x \to 0} \left(\frac{\tan x - x}{x^3} \cdot \frac{x}{\tan x}\right)$$

$$= \lim_{x \to 0} \frac{\tan x - x}{x^3}, \text{ since } \lim_{x \to 0} \frac{x}{\tan x} = 1$$

$$= \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x \to 0} \sec^2 x \left(\frac{\tan x}{x}\right)$$

$$= \frac{1}{3} \cdot 1.1 = \frac{1}{3}.$$

**Example 4.** *Evaluate* $\lim\limits_{x \to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$.

**Solution.** This is 0/0 form. So by L'Hospital's rule

$$\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)} = \lim_{x \to 0} \frac{e^x \cos x + e^x \sin x - 1 - 2x}{2x + \log(1-x) - x/(1-x)} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2e^x \cos x - 2}{2 - 1/(1-x) - 1/(1-x)^2} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2(e^x \cos x - e^x \sin x)}{-\{1/(1-x)^2\} - 2\{1/(1-x)^3\}}$$

$$= \frac{2}{-1-2} = -\frac{2}{3}.$$

**Example 5.** *Evaluate* $\lim\limits_{x \to 0} \dfrac{\tan^2 x - x^2}{x^2 \tan^2 x}$. [D.U., B.Sc. (G) 1998]

**Solution.** The given limit can be written as

$$\lim_{x \to 0} \frac{\tan^2 x - x^2}{x^4} \left(\frac{x^2}{\tan^2 x}\right) = \lim_{x \to 0} \frac{\tan^2 x - x^2}{x^4}, \text{ as } \lim_{x \to 0} \frac{x}{\tan x} = 1$$

$$= \lim_{x \to 0} \frac{2 \tan x \sec^2 x - 2x}{4x^3} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2 \sec^4 x + 4 \tan x \sec x (\sec x \tan x) - 2}{12 x^2}$$

$$= \lim_{x\to 0} \frac{\sec^4 x + 2\tan^2 x \sec^2 x - 1}{6x^2} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{4\sec^4 x \tan x + 4\tan x \sec^4 x + 4\tan^3 x \sec^2 x}{12x}$$

$$= \lim_{x\to 0} \left[\frac{2}{3}\sec^4 x\left(\frac{\tan x}{x}\right) + \frac{1}{3}\left(\frac{\tan x}{x}\right)\tan^2 x \,.\, \sec^2 x\right]$$

$$= \frac{2}{3}\,.\,1\,.\,1 + \frac{1}{3}\,.\,1\,.\,0\,.\,1 = \frac{2}{3}.$$

**Example 6.** *Evaluate* $\displaystyle \lim_{x\to 0} \frac{a^x - 1}{b^x - 1}$ $(b \neq 1)$. [D.U., B.A. (P) 1999, 92]

**Solution.** The given limit is 0/0 form and so

$$\lim_{x\to 0} \frac{a^x - 1}{b^x - 1} = \lim_{x\to 0} \frac{a^x \log a}{b^x \log b} = \frac{\log a}{\log b}.$$

**Example 7.** *For what value of a does*

$\displaystyle \frac{\sin 2x + a\sin x}{x^3}$ *tend to a finite limit as* $x \to 0$ ?

[D.U., Maths (H) 1992]

**Solution.** $\displaystyle \lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3} = \lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2} \left(\frac{2+a}{0}\right).$

To get the finite limit, we take $a + 2 = 0 \Rightarrow a = -2.$

Thus $\displaystyle \lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2} = \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2} \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6} = -1.$$

**Example 8.** *If* $\displaystyle \lim_{x\to 0} \frac{\sin 3x - a\sin x}{x^3}$ *is finite, find the value of a and the limit.*

[B.C.A. Sample Problem]

(**Ans.** $a = 3$)

**Please try yourself.**

**Examp**[illegible] **9.** [illegible]*etermine the values of p and q for which*

$$\lim_{x\to 0} \frac{x(1 + p\cos x) - q\sin x}{x^3}$$

*exists and equals* 1.

[D.U., Maths (H) 1997, 90 ; B.Sc. (G) 1988 ; B.A. (P) 1992]

**Solution.** It is a 0/0 form and so by L'Hospital's rule,

$$\lim_{x\to 0} \frac{x(1 + p\cos x) - q\sin x}{x^3}$$

$$= \lim_{x\to 0} \frac{(1 + p\cos x) - px\sin x - q\cos x}{3x^2}$$

$$= \lim_{x\to 0} \frac{1 + (p - q)\cos x - px\sin x}{3x^2} \left(\frac{1 + p - q}{0}\right)$$

If the required limit is to be 1, then $1 + p - q = 0$, for otherwise the limit will be infinite. Thus

$$\lim_{x\to 0} \frac{1 + (p - q)\cos x - px\sin x}{3x^2} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-(p - q)\sin x - p\sin x - px\cos x}{6x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-(p - q)\cos x - 2p\cos x + px\sin x}{6}$$

$$= \frac{-p + q - 2p}{6} = \frac{-3p + q}{6} = 1, \text{ as given}$$

$\Rightarrow \quad -3p + q = 6.$ Also $1 + p - q = 0$ and $-6 - 3p + q = 0.$

On solving these equations for $p$ and $q$, we obtain

$$p = -\tfrac{5}{2}, \; q = -\tfrac{3}{2}.$$

**[illegible]e 10.** *Find the values of a and b in order that*

$$\lim_{x\to 0} \frac{x(1 - a\cos x) + b\sin x}{x^3} = \frac{1}{3}.$$

[D.U., B.Sc. (G) 1999, 92 ; Maths (H) 1995]

**Solution.** The given limit is 0/0 form, so

$$\lim_{x\to 0} \frac{x(1 - a\cos x) + b\sin x}{x^3}$$

$$= \lim_{x\to 0} \frac{1 - a\cos x + ax\sin x + b\cos x}{3x^2} \left(\frac{1 - a + b}{0}\right)$$

The required limit will be $\frac{1}{3}$, provided $1 - a + b = 0.$

Assuming $1 - a + b = 0$ and applying L'Hospital's rule,

$$\lim_{x\to 0} \frac{x(1 - a\cos x) + b\sin x}{x^3}$$

$$= \lim_{x\to 0} \frac{(a - b)\sin x + a\sin x + ax\cos x}{6x}$$

$$= \lim_{x\to 0} \frac{(2a - b)\sin x + ax\cos x}{6x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{((2a-b)\cos x + a\cos x - ax\sin x}{6}$$

$$= \lim_{x\to 0} \frac{2a-b+a}{6} = \frac{3a-b}{6} = \frac{1}{3}, \text{ as given}$$

$\Rightarrow 3a-b=2,$ and $1-a+b=0.$

On solving these, we obtain $a=\frac{1}{2}, b=-\frac{1}{2}.$

**Example 11.** *Find the values of a, b, c so that*

$$\lim_{x\to 0} \frac{a+b\cos x + c\sin x}{x^2} \textit{ exists and equals } \tfrac{1}{2}.$$

**Solution.** The given limit is 0/0 form, if $a+b=0.$ ...(1)

Consequently, on applying L'Hospital's rule, the given limit is

$$\lim_{x\to 0} \frac{-b\sin x + c\cos x}{2x}, \text{ which is } \frac{0}{0} \text{ form, if } c=0.$$

Taking $c=0$, the above limit becomes

$$= \lim_{x\to 0} \frac{-b\sin x}{2x} = -\frac{b}{2} \lim_{x\to 0} \frac{\sin x}{x} = \frac{b}{2}\cdot 1 = -\frac{b}{2} = \frac{1}{2}, \text{ as given.}$$

Hence $b=-1, c=0, a=1$ ; by (1).

**Example 12.** *Find a and b such that* $\lim_{x\to 0} \dfrac{ae^x + be^{-x} + 2\sin x}{\sin x + x\cos x} = 2.$

**Solution.** The given limit is 0/0 form, if $a+b=0.$

Consequently, on applying L'Hospital's rule, the given limit is

$$= \lim_{x\to 0} \frac{ae^x - be^{-x} + 2\cos x}{2\cos x - x\sin x} = \frac{a-b+2}{2} = 2, \text{ as given}$$

$\Rightarrow a-b=2$ and $a+b=0.$

Hence $a=1$ and $b=-1.$

**Example 13.** *Find the values a, b, c so that*

$$\lim_{x\to 0} \frac{ae^x - b\cos x + c\,e^{-x}}{x\sin x}$$

*may be equal to 2.* [D.U., B.C.A. 1999]

**Solution.** The given limit is 0/0 form, if $a-b+c=0.$ ...(1)

Consequently, on applying L'Hospital's rule, given limit is

$$= \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x}, \text{ which is } \frac{0}{0} \text{ form, if } a-c=0. \quad ...(2)$$

$$= \lim_{x\to 0} \frac{a\,e^x + b\cos x + c\,e^{-x}}{2\cos x - x\sin x} = \frac{1}{2}(a+b+c). \quad ...(3)$$

It is given that $\frac{1}{2}(a+b+c)=2$ or $a+b+c=4.$

Subtracting (1) from (3), $2b=4$ or $b=2$

Now (3) becomes $a+c=2$. From (2), $a=c$

Hence $a=c=1$ and $b=2.$

**Example 14.** *Find the value of α so that*

$$\lim_{x\to 0} \frac{e^{\alpha x} - e^x - x}{x^2} = \frac{3}{2}.$$

[B.C.A. Sample Problem]

**Solution.** The given limit is 0/0 form and so by L'Hospital's rule it is

$$= \lim_{x\to 0} \frac{\alpha\,e^{\alpha x} - e^x - 1}{2x}, \text{ which is } \frac{0}{0} \text{ form, if } \alpha-2=0 \text{ or } \alpha=2.$$

Taking $\alpha=2$, the above limit becomes

$$\lim_{x\to 0} \frac{2\,e^{2x} - e^x - 1}{2x} = \lim_{x\to 0} \frac{4\,e^{2x} - e^x}{2} = \frac{3}{2}$$

**Example 15.** *Evaluate* $\lim_{x\to 0} \dfrac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$ [D.U., B.C.A. 1998]

**Solution.** Since $\lim_{x\to 0} (1+x)^{1/x} = e$, the given limit is 0/0 form.

Before we evaluate the limit, we find the expansion of $(1+x)^{1/x}$.

Let $y=(1+x)^{1/x}$, so that

$$\log y = \frac{1}{x}\log(1+x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right)$$

$$= \left(1 - \frac{x}{2} + \frac{x^2}{3} - \dots\right) = 1+\alpha, \text{ where } \alpha = -\frac{x}{2} + \frac{x^2}{3} - \dots\dots$$

Now $\log y = 1+\alpha \Rightarrow y = e^{1+\alpha} = e.e^{\alpha} \Rightarrow y = e\left(1+\alpha+\dfrac{\alpha^2}{2!}+\dots\dots\right)$

$$y = e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-\dots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\dots\right)^2+\dots\right]$$

$$(1+x)^{1/x} = e\left[1-\frac{x}{2}+\frac{11}{24}x^2-\dots\right] \quad ...(A)$$

Now
$$\lim_{x\to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$$

$$= \lim_{x\to 0} \frac{e\left(1-\frac{x}{2}+\frac{11}{24}x^2-\dots\right) - e + \frac{1}{2}ex}{x^2}$$

$$= \lim_{x\to 0}\left(\frac{11\,e}{24} + \text{terms containing } x\right) = \frac{11\,e}{24}.$$

**Example 16.** *Find* $\lim\limits_{x \to 0} \dfrac{(1+x)^{1/x} - e}{x}$.

**Solution.** Using (A) of Example 15, we obtain

$$\therefore \quad \lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x} = \lim_{x \to 0} \frac{e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - \dots\right) - e}{x}$$

$$= \lim_{x \to 0} \left(-\frac{e}{2} + \text{terms containing } x\right) = -\frac{e}{2}.$$

**Example 17.** *Evaluate* $\lim\limits_{x \to \frac{1}{2}\pi} \dfrac{\tan 3x}{\tan x}$. [D.U., Maths (H) 1994]

**Solution.** The given limit $\infty/\infty$ form. By L' Hospital's rule for $\infty/\infty$ form, we see that

$$\lim_{x \to \frac{1}{2}\pi} \frac{\tan 3x}{\tan x} = \lim_{x \to \frac{1}{2}\pi} \frac{3 \sec^2 3x}{\sec^2 x} = \lim_{x \to \frac{1}{2}\pi} \frac{3 \cos^2 x}{\cos^2 3x}\left(\frac{0}{0}\right)$$

$$= \lim_{x \to \frac{1}{2}\pi} \frac{-6 \cos x \sin x}{-6 \cos 3x \sin 3x} = \lim_{x \to \frac{1}{2}\pi} \frac{\sin 2x}{\sin 6x}\left(\frac{0}{0}\right)$$

$$= \lim_{x \to \frac{1}{2}\pi} \frac{2 \cos 2x}{6 \cos 6x} = \frac{1}{3}.$$

**Example 18.** *Evaluate* $\lim\limits_{x \to 0+0} \dfrac{\cot x}{\log \sin x}$.

**Solution.** The given limit is $\infty/\infty$ form. By L'Hospital's rule for $\infty/\infty$ form, we have

$$\lim_{x \to 0+0} \frac{\cot x}{\log \sin x} = \lim_{x \to 0+0} \frac{-\operatorname{cosec}^2 x}{\cot x}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0+0} \frac{2 \operatorname{cosec}^2 x \cot x}{-\operatorname{cosec}^2 x} = \lim_{x \to 0+0} (-2 \cot x) = -\infty.$$

**Example 19.** *Evaluate* $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$.

[D.U., B.Sc. (G) 1995, 91 ; B.A. (P) 1994 ; Maths (H) 1999]

**Solution.** $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$ is $\infty - \infty$ form. Such a form can be reduced to 0/0 or $\infty/\infty$ form. Now

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x \to 0} \frac{\sin x - x}{x \sin x} = \lim_{x \to 0} \frac{\sin x - x}{x^2}\left(\frac{x}{\sin x}\right)$$

$$= \lim_{x \to 0} \frac{\sin x - x}{x^2}\left(\frac{0}{0}\right), \qquad \because \lim_{x \to 0} \frac{x}{\sin x} = 1$$

$$= \lim_{x \to 0} \frac{\cos x - 1}{2x}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{-\sin x}{2} = 0.$$

**Example 20.** *Find the value of* $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$.

[D.U., Maths (H) 1989 ; B.Sc (G) 1997 ; B.A. (P) 1991]

**Solution.** The given limit is $\infty - \infty$ form. Now

$$\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = \lim_{x \to 0} \left(\frac{\sin^2 x - x^2}{x^2 \sin^2 x}\right) = \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4}\left(\frac{x}{\sin x}\right)^2$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{2 \sin x \cos - 2x}{4x^3}$$

$$= \lim_{x \to 0} \frac{\sin 2x - 2x}{4x^3}\left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{2 \cos 2x - 2}{12x^2}$$

$$= \lim_{x \to 0} \frac{\cos 2x - 1}{6x^2} = \lim_{x \to 0} \frac{(1 - 2 \sin^2 x) - 1}{6x^2}$$

$$= -\frac{1}{3} \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 = -\frac{1}{3}.$$

**Example 21.** *Find* $\lim\limits_{x \to \frac{1}{2}\pi} (\sec x - \tan x)$.

**Solution.** The given limit is $\infty - \infty$ form. We have

$$\lim_{x \to \frac{1}{2}\pi} (\sec x - \tan x) = \lim_{x \to \frac{1}{2}\pi} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x \to \frac{1}{2}\pi} \frac{1 - \sin x}{\cos x}\left(\frac{0}{0}\right) = \lim_{x \to \frac{1}{2}\pi} \frac{-\cos x}{-\sin x} = \frac{0}{1} = 0.$$

**Example 22.** *Evaluate* $\lim\limits_{x \to 0} \left(\dfrac{1}{e^x - 1} - \dfrac{1}{x}\right)$.

[D.U., B.A. (P) 1999 ; Maths (H) 1998 ; B.Sc. (G) 1990]

**Solution.** The given limit is $\infty - \infty$ form. We have

$$\lim_{x \to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) = \lim_{x \to 0} \frac{x - (e^x - 1)}{x(e^x - 1)}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{1-e^x}{e^x - 1 + x e^x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-e^x}{e^x + x e^x + e^x} = -\frac{1}{2}.$$

**Example 23.** *Evaluate* $\lim\limits_{x\to 0} \left(\frac{1}{x} - \cot x\right)$.

[D.U., B.A. (P) 1997 ; B.Sc. (G) 1996, 93]

**Solution.** The given limit is $\infty - \infty$ form. We have

$$\lim_{x\to 0} \left(\frac{1}{x} - \cot x\right) = \lim_{x\to 0} \left(\frac{\sin x - x\cos x}{x \sin x}\right).$$

$$= \lim_{x\to 0} \frac{\sin x - x\cos x}{x^2}, \text{ as } \lim_{x\to 0} \frac{x}{\sin x} = 1.$$

$$= \lim_{x\to 0} \frac{\cos x - \cos x + x\sin x}{2x} = \lim_{x\to 0} \frac{\sin x}{2} = 0.$$

**Example 24.** *Evaluate* $\lim\limits_{x\to 4} \left[\frac{1}{\log(x-3)} - \frac{1}{x-4}\right]$.

**Solution.** The given limit is $\infty - \infty$ form. We have

$$\lim_{x\to 4} \left[\frac{1}{\log(x-3)} - \frac{1}{x-4}\right]$$

$$= \lim_{x\to 4} \left[\frac{x-4-\log(x-3)}{(x-4)\log(x-3)}\right] \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 4} \frac{1 - 1/(x-3)}{\log(x-3) + (x-4)/(x-3)} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 4} \frac{1/(x-3)^2}{1/(x-3) + 1/(x-3)^2} = \frac{1}{1+1} = \frac{1}{2}.$$

**Example 25.** *Evaluate* $\lim\limits_{x\to\infty} x(a^{1/x} - 1), a > 0$.

**Solution.** Let $x = \frac{1}{t}$, so that $t = \frac{1}{x}$. $\therefore$ $x \to \infty \Leftrightarrow t \to 0$. Now

$$\lim_{x\to\infty} x(a^{1/x} - 1) = \lim_{t\to 0} \frac{a^t - 1}{t} \left(\frac{0}{0}\right) = \lim_{t\to 0} \frac{a^t \log_e a}{1} = \log_e a.$$

**Example 26.** *Evaluate* $\lim\limits_{x\to 1} (1-x) \tan\left(\frac{1}{2}\pi x\right)$.

[D.U., B.Sc. (G) 1998]

**Solution.** The given limit is $0 . \infty$ form. We can write it as

$$\lim_{x\to 1} \frac{(1-x)}{\cot\left(\frac{\pi}{2}x\right)} \left(\frac{0}{0}\right) = \lim_{x\to 1} \frac{-1}{-\frac{\pi}{2}\operatorname{cosec}^2\left(\frac{\pi}{2}x\right)} = \frac{2}{\pi}.$$

**Ex**[illegible]**7.** *Evaluate* $\lim\limits_{x\to 1} \sec\left(\frac{\pi}{2}x\right) \log\left(\frac{1}{x}\right)$.

**Solution.** The given limit is $0 . \infty$ form. We can write it as

$$\lim_{x\to 1} \frac{\log(1/x)}{\cos\left(\frac{\pi}{2}x\right)} \left(\frac{0}{0}\right) = \lim_{x\to 1} \frac{x\left(-\frac{1}{x^2}\right)}{-\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)} = \frac{2}{\pi}.$$

**Example 28.** *Evaluate* $\lim\limits_{x\to 0} \frac{\log\log(1-x^2)}{\log\log\cos x}$. [D.U., B.C.A. 1999]

**Solution.** The given limit is $\infty/\infty$ form and so by L'Hospital's rule it

$$= \lim_{x\to 0} \frac{-2x \,.\, \cos x \,.\, \log\cos x}{(1-x^2)\log(1-x^2)(-\sin x)} = \lim_{x\to 0} \frac{2\log\cos x}{(1-x^2)\log(1-x^2)} \left(\frac{0}{0}\right)$$

$$\left[\because \lim_{x\to 0} (x/\sin x) = 1 \text{ and } \lim_{x\to 0} \cos x = 1\right].$$

$$= \lim_{x\to 0} \frac{-2\sin x}{\cos x\,[(1-x^2)\{-2x/(1-x^2)\} - 2x\log(1-x^2)]}$$

$$\lim_{x\to 0} \frac{\tan x}{x[1+\log(1-x^2)]} = \lim_{x\to 0} \frac{1}{1+\log(1-x^2)} = 1.$$

$$\left[\because \lim_{x\to 0} (\tan x/x) = 1\right]$$

**Example 29.** *Evaluate* $\lim\limits_{x\to 1-0} \frac{\log(1-x)}{\cot(\pi x)}$. [D.U., Maths (H) 1996]

**Solution.** Given limit is $\infty/\infty$ form and so it is

$$= \lim_{x\to 1-0} \frac{-1/(1-x)}{-\pi\operatorname{cosec}^2(\pi x)} = \lim_{x\to 1-0} \frac{\sin^2(\pi x)}{\pi(1-x)} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1-0} \frac{2\pi\sin(\pi x)\cos(\pi x)}{-\pi} = 0.$$

**Example 30.** *Evaluate* $\lim\limits_{x\to 0} \left(\cot^2 x - \frac{1}{x^2}\right)$.

[D.U., B.Sc. (G) 1997 ; Maths (H) 1991 ; B.A. (P) 1990]

## 9.4 The Indeterminate Forms $0^0$, $1^\infty$ and $\infty^0$

To evaluate $\lim\limits_{x \to a} [f(x)]^{g(x)}$,

where $\lim\limits_{x \to a} f(x) = 0, 1$ or $\infty$ and $\lim\limits_{x \to a} g(x) = 0, \infty$ or 0.

We write $y = [f(x)]^{g(x)}$

$\Rightarrow \quad \log y = g(x) \log f(x)$

$\Rightarrow \quad \lim\limits_{x \to a} \log y = \lim\limits_{x \to a} [g(x) . \log f(x)].$

The R.H.S. can be reduced to the form 0/0 or ∞/∞, which can be evaluated. Suppose $\lim\limits_{x \to a} [g(x) . \log f(x)] = l.$

Then $\lim\limits_{x \to a} \log y = l \Rightarrow \log \lim\limits_{x \to a} y = l \Rightarrow \lim\limits_{x \to a} y = e^l.$

Hence $\lim\limits_{x \to a} [f(x)]^{g(x)} = e^l.$

### EXAMPLES

**Example 31.** *Evaluate* $\lim\limits_{x \to \frac{1}{2}\pi} (\sin x)^{\tan x}$.

**Solution.** Let $y = (\sin x)^{\tan x}$ . $(1^\infty)$

$\therefore \quad \log y = \tan x \log \sin x$

$\Rightarrow \quad \lim\limits_{x \to \frac{1}{2}\pi} y = \lim\limits_{x \to \frac{1}{2}\pi} (\tan x \log \sin x) \quad (\infty . 0)$

$$= \lim_{x \to \frac{1}{2}\pi} \frac{\log \sin x}{\cot x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \frac{1}{2}\pi} \frac{\cos x}{\sin x(-\operatorname{cosec}^2 x)} = 0.$$

$\therefore \quad \lim\limits_{x \to \frac{1}{2}\pi} \log y = 0 \Rightarrow \lim\limits_{x \to \frac{1}{2}\pi} y = e^0 = 1.$

Hence $\lim\limits_{x \to \frac{1}{2}\pi} (\sin x)^{\tan x} = 1.$

**Example 32.** *Evaluate* $\lim\limits_{x \to 0} (\cot x)^{\frac{1}{\log x}}$.

**Solution.** Let $y = (\cot x)^{\frac{1}{\log x}} \quad (\infty^0)$

$$= \lim_{x \to 0} \frac{x \sec^2 x - \tan x}{2x^2 \tan x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\sec^2 x + 2x \sec^2 x \tan x - \sec^2 x}{4x \tan x + 2x^2 \sec^2 x}$$

$$= \lim_{x \to 0} \frac{\sec^2 x \tan x}{2 \tan x + x \sec^2 x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\sec^4 x + 2 \sec x (\sec x \tan x) \tan x}{2 \sec^2 x + \sec^2 x + 2x \sec^2 x \tan x} = \frac{1}{3}.$$

$$\therefore \quad \lim_{x \to 0} \log y = \frac{1}{3} \Rightarrow \lim_{x \to 0} y = e^{1/3}.$$

Hence $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$.

**Example 35.** *Evaluate* $\lim_{x \to \frac{1}{2}\pi} (\sin x)^{\tan^2 x}$.

[D.U., B.Sc. (G) 1991 ; B.A. (P) 1996]

**Solution.** The given limit is $1^\infty$ form.

Let $y = (\sin x)^{\tan^2 x}$, so that $\log y = \tan^2 x \log \sin x$.

$$\therefore \quad \lim_{x \to \frac{1}{2}\pi} \log y = \lim_{x \to \frac{1}{2}\pi} \tan^2 x \log \sin x = \lim_{x \to \frac{1}{2}\pi} \frac{\log \sin x}{\cot^2 x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \frac{1}{2}\pi} \frac{\cot x}{-2 \cot x \operatorname{cosec}^2 x} = -\frac{1}{2}.$$

$$\therefore \quad \lim_{x \to \frac{1}{2}\pi} \log y = \log \lim_{x \to \frac{1}{2}\pi} (\sin x)^{\tan^2 x} = -\frac{1}{2}.$$

Hence $\lim_{x \to \frac{1}{2}\pi} (\sin x)^{\tan^2 x} = e^{-1/2}$.

**Exmaple 36.** *Evaluate* $\lim_{x \to 0} (1 + x)^{1/x}$. [D.U., B.Sc. (G) 1995]

**Solution.** Let $y = (1 + x)^{1/x}$, so that $\log y = \frac{1}{x} \log (1 + x)$.

$$\therefore \quad \lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log (1 + x)}{x} \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{1/(1 + x)}{1} = 1.$$

$$\therefore \quad \lim_{x \to 0} \log y = \log \lim_{x \to 0} y = 1$$

Hence $\lim_{x \to 0} (1 + x)^{1/x} = e^1 = e$.

**Example 37.** *Evaluate* $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^x$. [D.U., B.C.A. 1999]

**Solution.** Letting $t = 3/x$, we obtain

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \lim_{t\to 0}\,(1+t)^{3/t}.$$

Let $y = (1+t)^{3/t}$, so that $\log y = \dfrac{3}{t}\log(1+t)$.

$$\therefore \quad \lim_{t\to 0}\log y = \lim_{t\to 0}\frac{3\log(1+t)}{t}\left(\frac{0}{0}\right) = \lim_{t\to 0}\frac{3}{(1+t)\,.\,1} = 3.$$

$$\therefore \quad \log\lim_{t\to 0} y = 3 \quad\text{or}\quad \lim_{t\to 0} y = e^3 \quad\text{or}\quad \lim_{t\to 0}\,(1+t)^{3/t} = e^3.$$

Hence $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^x = e^3$.

**Example 38.** *Show that* $\lim\limits_{x\to 0}\,(1+x)^{2/x} = e^2$.

**Please try yourself.**

**Example 39.** *Evaluate* $\lim\limits_{x\to\frac{1}{2}\pi-0}\,(\tan x)^{\sin 2x}$. [D.U., Maths (H) 1999]

**Solution.** The given limit is $\infty^0$ form.

Let $y = (\tan x)^{\sin 2x}$, so that $\log y = \sin 2x\log\tan x$.

$$\therefore \quad \lim_{x\to\frac{1}{2}\pi-0}\log y = \lim_{x\to\frac{1}{2}\pi-0}\frac{\log\tan x}{\operatorname{cosec} 2x}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\frac{1}{2}\pi-0}\frac{\sec^2 x}{\tan x\,(-2\operatorname{cosec} 2x\cot 2x)}$$

$$= \lim_{x\to\frac{1}{2}\pi-0}\frac{\sin^2 2x}{-2\sin x\cos x\,.\,\cos 2x}$$

$$= \lim_{x\to\frac{1}{2}\pi-0}\left(\frac{-\sin 2x}{\cos 2x}\right) = 0.$$

Hence $\lim\limits_{x\to\frac{1}{2}\pi-0} y = \lim\limits_{x\to\frac{1}{2}\pi-0}\,(\tan x)^{\sin 2x} = e^0 = 1$.

**Example 40.** *Evaluate* $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$.

**Solution.** The given limit is $1^\infty$ form.

Let $y = \left(\dfrac{\sin x}{x}\right)^{1/x^2}$, so that $\log y = \dfrac{1}{x^2}\log\left(\dfrac{\sin x}{x}\right)$.

$$\therefore \quad \lim_{x\to 0}\log y = \lim_{x\to 0}\frac{\log(\sin x/x)}{x^2}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\dfrac{x}{\sin x}\left(\dfrac{x\cos x-\sin x}{x^2}\right)}{2x} = \lim_{x\to 0}\frac{x\cos x-\sin x}{2x^2\sin x}$$

$$= \lim_{x\to 0}\frac{x\cos x-\sin x}{2x^3}, \quad\text{as}\quad \lim_{x\to 0}\frac{x}{\sin x} = 1$$

$$= \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{6x^2} = \lim_{x\to 0} -\frac{\sin x}{6x} = -\frac{1}{6}.$$

$$\therefore \quad \lim_{x\to 0}\log y = \log\lim_{x\to 0} y = -\frac{1}{6}.$$

Hence $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$.

**Example 41.** *Evaluate* $\lim\limits_{x\to a}\left(2-\dfrac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$.

**Solution.** The given limit is $1^\infty$ form.

Let $y = \left(2-\dfrac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$. Then $\log y = \tan\left(\dfrac{\pi x}{2a}\right)\log\left(2-\dfrac{x}{a}\right)$

$$\therefore \quad \lim_{x\to a}\log y = \lim_{x\to a}\frac{\log\left(2-\dfrac{x}{a}\right)}{\cot(\pi x/2a)}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to a}\frac{-\dfrac{1}{a}\cdot\dfrac{1}{(2-x/a)}}{-(\pi/2a)\operatorname{cosec}^2(\pi x/2a)} = \frac{2}{\pi}.$$

$\therefore \quad \log\lim\limits_{x\to a} y = \dfrac{2}{\pi}$. Hence $\lim\limits_{x\to a} y = e^{2/\pi}$.

**Example 42.** *Evaluate* $\lim\limits_{x\to\pi/4}\,(\tan x)^{\tan 2x}$.

**Solution.** The given limit is $1^\infty$ form.

**Example 45.** *Evaluate* $\lim\limits_{x \to \frac{\pi}{2}-0} \left(\frac{\pi}{2} - x\right)^{\tan x}$ [D.U., Maths (H) 1994]

**Solution.** Let $y = \left(\frac{\pi}{2} - x\right)^{\tan x}$, so that $\log y = \tan x \,.\, \log\left(\frac{\pi}{2} - x\right)$.

$$\therefore \quad \lim_{x \to \frac{\pi}{2}-} \log y = \lim_{x \to \frac{\pi}{2}-} \tan x \,.\, \log\left(\frac{\pi}{2} - x\right) = \infty \,.\, \infty = \infty$$

or $\quad \log \lim\limits_{x \to \frac{\pi}{2}-} y = \infty.\quad$ Hence $\lim\limits_{x \to \frac{\pi}{2}-} y = 0. \qquad (\because \log 0 = \infty)$

**Example 46.** $\lim\limits_{x \to 0} \left(\frac{1}{x}\right)^{\tan x}$. [D.U., B.Sc. (G) 1994]

**Solution.** Let $y = (1/x)^{\tan x}$, so that $\log y = \tan x \log (1/x)$.

$$\therefore \quad \lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log(1/x)}{\cot x} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{x(-1/x^2)}{-\operatorname{cosec}^2 x} = \lim_{x \to 0} \frac{\sin^2 x}{x}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 . x = 1 \times 0 = 0.$$

Hence $\quad \lim\limits_{x \to 0} y = e^0 = 1.$

**Example 47.** $\lim\limits_{x \to 0+} (\cos x)^{1/x^3}$.

**Solution.** Let $y = (\cos x)^{1/x^3}$. Then $\log y = \frac{1}{x^3} \log \cos x$.

$$\therefore \quad \lim_{x \to 0+} \log y = \lim_{x \to 0+} \frac{\log \cos x}{x^3} = \lim_{x \to 0+} \frac{-\tan x}{x} \cdot \frac{1}{3x} = \infty.$$

Hence $\log \lim\limits_{x \to 0+} y = \infty \Rightarrow \lim\limits_{x \to 0+} y = 0 \Rightarrow \lim\limits_{x \to 0+} (\cos x)^{1/x^3} = 0.$

**Example 48.** *Evaluate* $\lim\limits_{x \to 0} (1 + \sin x)^{\cot x}$.

**Solution.** The given limit is $1^\infty$ form.

Let $\quad y = (1 + \sin x)^{\cot x}$, so that $\log y = \cot x \log (1 + \sin x)$.

$$\therefore \quad \lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log (1 + \sin x)}{\tan x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \infty} \frac{-2x(1+x^2)}{(1+x^2)^2} = \lim_{x \to \infty} \frac{-2x}{1+x^2} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{-2}{2x} = 0.$$

$$\therefore \quad \log \lim_{x \to \infty} y = 0 \Rightarrow \lim_{x \to \infty} y = e^0 = 1.$$

**Hence** $\lim_{x \to \infty} \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x} = 1.$

## EXERCISES

Evaluate each of the following limits :

1. $\lim_{x \to 0} \frac{x \cos x - \log(1+x)}{x^2}$. [D.U., B.Sc. (G) 1994 ; B.A. (P) 90] **(Ans.** $\frac{1}{2}$**)**

2. $\lim_{x \to 0} \frac{\log(1+x) - x}{1 - x \cos x}$. [D.U., B.A. (P) 1999] **(Ans. 0)**

3. $\lim_{x \to 0} \frac{x - \sin x}{x^3}$. **(Ans.** $\frac{1}{6}$**)**

4. $\lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin x}$. **(Ans. 1)**

[**Hint.** Use $\cosh 0 = 1$]

5. $\lim_{x \to 0} \frac{\tan x - x}{\sin x - x}$. **(Ans. – 2)**

6. $\lim_{x \to 0+0} \frac{\log x}{\operatorname{cosec} x}$. **(Ans. 0)**

7. What is wrong with the following use of L'Hospital's rule :

$$\lim_{x \to 0} \frac{x^4 - 4x^3 + 3}{3x^2 - x - 2} = \lim_{x \to 1} \frac{4x^3 - 12x^2}{6x - 1} = \lim_{x \to 1} \frac{12x^2 - 24x}{6} = -2\ ?$$

[**Hint.** $\frac{4x^3 - 12x^2}{6x - 1}$ is not $\frac{0}{0}$ form (In fact, this limit is $-\frac{8}{5}$)

and so we cannot apply L' Hospital's rule further]

8. $\lim_{x \to 0} \frac{e^{ax} - e^{-ax}}{\log(1+bx)}$. **(Ans.** $2a/b$**)**

9. $\lim\limits_{x \to \frac{1}{2}\pi + 0} \dfrac{\log (x - \frac{1}{2}\pi)}{\tan x}$. (**Ans.** 0)

10. $\lim\limits_{x \to 0} \dfrac{e^x - 2\cos x + e^{-x}}{x \sin x}$. (**Ans.** 2)

11. $\lim\limits_{x \to 1-0} \dfrac{\log (1-x)}{\cot (\pi x)}$. (**Ans.** 0)

12. $\lim\limits_{x \to 0+0} \dfrac{\log \tan 2x}{\log \tan x}$. (**Ans.** 1)

13. $\lim\limits_{x \to \frac{1}{2}\pi} \left( \sec x - \dfrac{1}{1 - \sin x} \right)$ (**Ans.** $\infty$)

[D.U., B.A. (P) 1995]

14. $\lim\limits_{x \to 0} (\operatorname{cosec} x - \cot x)$.

[**Hint.** Given limit $= \lim\limits_{x \to 0} \dfrac{1 - \cos x}{\sin x} = \lim\limits_{x \to 0} \dfrac{\sin x}{\cos x} = 0.$]

15. $\lim\limits_{x \to 1} \left( \dfrac{x}{x-1} - \dfrac{1}{\log x} \right)$. (**Ans.** $\frac{1}{2}$)

16. $\lim\limits_{x \to \frac{1}{2}\pi} (1 - \sin x) \tan x$. (**Ans.** 0 )

17. $\lim\limits_{x \to 0+0} x^x$. (**Ans.** 1)

18. $\lim\limits_{x \to 1} (2-x)^{\tan (\pi x/2)}$. (**Ans.** $e^{2/\pi}$)

[**Hint.** Take $a = 1$ in Example 41]

19. $\lim\limits_{x \to 0} (\cos x)^{1/x}$. (**Ans.** 1)

20. $\lim\limits_{x \to 0+0} \left( \log \dfrac{1}{x} \right)^x$. (**Ans.** 1)

21. $\lim\limits_{x \to 0} \dfrac{\sin x - x + \frac{1}{6}x^3}{x^5}$. (**Ans.** $\frac{1}{120}$)

[**Hint.** Use $\sin x = x - \dfrac{x^3}{3\,!} + \dfrac{x^5}{5\,!} - \dfrac{x^7}{7\,!} + \ldots$]

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# APPENDIX

## DELHI UNIVERSITY EXAMINATION PAPERS

B.A. /B.Sc. (Hons.)/I – 1992
MATHEMATICS – Unit III
(Analysis—I)

*hours* *Maximum Marks* : 50

*Attempt **One** question only from each Section.*

### SECTION I

Define modulus of a real number. If $x, y$ are any real numbers, prove that : $|x + y| \leq |x| + |y|$.

What is a closed set ? Prove that the intersection of any arbitrary family of closed sets is closed.

Prove that the union of any arbitrary family of open sets is open.

If $x, y$ are any real numbers, prove that $||x| - |y|| \leq |x - y|$.

State and prove Bolzano-Weierstrass Theorem on sets.

What is a closed set ? Prove that the union of any finite number of closed set is closed.

### SECTION II

Prove that a necessary and sufficient condition for a monotonic sequence to be convergent is that it is bounded.

Show by applying Cauchy's convergence criterion, that the sequence $< a_n >$ defined by

$$a_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \text{ does not converge.}$$

Prove that if $p > 0$, then $\lim\limits_{n \to \infty} \dfrac{n^k}{(1+p)^n} = 0$, $k$ being a fixed real number.

Prove that a real number $p$ is a cluster point of a sequence $< a_n >$ if and only if there exists a subsequence of $< a_n >$ converging to $p$.

(b) Prove that the sequence $< a_n >$ recursively defined by $a_1 = \sqrt{5}$, $a_{n+1} = \sqrt{5 + a_n}$, $n \geq 1$ converges to the positive root of the equation $x^2 - x - 5 = 0$.

### SECTION III

(a) State and prove Leibnitz test for the convergence of an alternating series.

(b) Test for convergence the following series :

(*i*) $\dfrac{x}{1} + \dfrac{1}{2} \cdot \dfrac{x^2}{3} + \dfrac{1\,.\,3}{2\,.\,4} \cdot \dfrac{x^3}{5} + \dfrac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6} \cdot \dfrac{x^4}{7} + \ldots\ (x > 0)$.

(*ii*) $\sum\limits_{n=1}^{\infty} \left( 1 + \dfrac{1}{\sqrt{n}} \right)^{-n^{3/2}}$.

## SECTION III

3. (*a*) (*i*) Obtain the Maclaurin's series expansion of $e^x$.

(*ii*) For what value of $a$ does $\dfrac{\sin 2x + a \sin x}{x^3}$ tend to a finite limit as $x \to 0$ ?

(b) Prove that $\dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1$, whenever $0 < x < \dfrac{\pi}{2}$.

(*c*) Find the maxima and minima of the function $f$ defined by

$$f(x) = \sin x + \frac{1}{2} \sin 2x + \frac{1}{2} \sin 3x, \text{ for all } x \in [0, \pi].$$

## SECTION IV (Integral Calculus)

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### B.A./B.Sc. (Hons.)/I – 1993
### MATHEMATICS – Unit III
### (Analysis—I)

*Time : 2 hours* *Maximum Marks : 50*

*Attempt* ***four*** *questions in all,*
*selecting only* **one** *question from each Section.*

## SECTION I

1. (*a*) Define neighbourhood of a point, open set and limit point of a set. Prove that the intersection of two open sets is open.

Does the above result hold for the intersection of an arbitrary family of open sets ? Illustrate your answer by an example.

(*b*) State and prove Bolzano-Weierstrass Theorem for sets.

2. (*a*) Define open set, closed set in **R**. Prove that the union of two closed sets is closed. Does the above result hold for the union of an arbitrary family of closed sets ? Illustrate your answer by an example.

(*b*) Prove that the intersection of any arbitrary family of closed sets is closed.

(*c*) State Bolzano-Weierstrass theorem (without proof). Show by an example that condition of boundedness cannot be relaxed.

## SECTION II

3. (*a*) If a sequence $< a_n >$ in **R** converges to $l \in$ **R**, prove that the sequence $< \dfrac{a_1 + a_2 + \ldots + a_n}{n} >$ also converges to $l$.

(*b*) A sequence $< a_n >$ is defined as follows :

$$a_1 = 1, a_{n+1} = \frac{4 + 3a_n}{3 + 2a_n}, n \geq 1.$$

Show that the sequence $< a_n >$ converges and find its limit.

## SECTION II

2. (*a*) Let $f$ and $g$ be two functions having the same domain $D$. If $fg$ is derivable at $x_0 \in D$, is it necessary that $f$ and $g$ are both derivable at $x_0$? Give argument in support of your answer.

(*b*) Let $f$ be defined on **R** by setting

$$f(x) = x\frac{e^{-1/x} - e^{1/x}}{e^{-1/x} + e^{1/x}}, \quad \text{if } x \neq 0$$

$$= 0, \quad \text{if } x = 0.$$

Show that $f$ is not derivable at $x = 0$.

(*c*) Let $f$ be defined and continuous on $[a-h, a+h]$ and derivable on $]a-h, a+h[$. Prove that there is a real number $\theta$ between 0 and 1 for which

$$f(a+h) - 2f(a) + f(a-h) = h\{f'(a+\theta h) - f'(a-\theta h)\}.$$

## SECTION III

3. (*a*) (*i*) Obtain the Maclaurin's series expansion of $\sin x$.

(*ii*) Evaluate $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^2 \tan x}$.

(*b*) Applying Lagrange's mean value theorem to the function defined by $f(x) = \log(1+x)$ for all $x \geq 0$, show that $0 < [\log(1+x)]^{-1} - x^{-1} < 1$, whenever $x > 0$.

(*c*) Let $f$ be derivable on an interval $]c-\varepsilon, c+\varepsilon[$, and let $f'(c) = 0$. Prove that if

(*i*) $f''(c) < 0$, then $f$ has a maximum at $x = c$,

(*ii*) $f''(c) > 0$, then $f$ has a minimum at $x = c$.

## SECTION IV (Integral Calculus)

4. *(a)* Let $f$ be continuous on a closed interval $[a, b]$. Show that Sup $f$ and Inf $f$ are attained in $[a, b]$.

*(b)* Let $f$ be continuous over $[a, b]$ such that $f(a)f(b) < 0$. Show there exists at least one $c \in ]\, a, b\,[$ such that $f(c) = 0$.

*(c)* Show $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty, [, a > 0$.

5. *(a)* State and prove Lagrange's Mean value theorem.

*(b)* Obtain Maclaurin's series expansion of $f(x) = \log(1 + x)$, $|x| < 1$.

*(c)* Verify Lagrange's mean value theorem for $f(x) = \sqrt{x^2 - 4}$ in $[2, 4]$.

6. *(a)* Prove that :

$$0 < [\log(1 + x)]^{-1} - x^{-1} < 1, \forall x > 0.$$

*(b)* Evaluate :

*(i)* $\underset{x \to 0}{\text{Lt}} \frac{x \cos x - \log(1 + x)}{x^2}$; *(ii)* $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x}\right)^{\tan x}$

*(c)* Show that $x^5 - 5x^4 + 5x^3 - 10$ has a maximum at 1, a minimum at 3 and neither at 0.. Also find maximum and minimum values.

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### B.A. (P) /III—1994

### MATHEMATICS—Paper III (ii)

(Statistics and Real Analysis)

*Time : 3 hours* *Maximum Marks : 100*

All Sections are compulsory and have equal marks.

Attempt any two parts from each Section.

### Part A (Real Analysis)

### Section 1

1. (a) Define neighbourhood of a point, open set and limit point of a set. Prove that intersection of two open sets is open.

*(b)* State Bolzano Weierstrass Theorem (without proof). Show by an example that condition of boundedness cannot be relaxed.

(c) Define uniform continuity. Prove that every uniformly continuous function is continuous. It the converse true ? Justify your answer.

*(d)* Let $f$ and $g$ be defined on an interval $[a, b]$. If $f$ and $g$ are continuous at $x_0 \in [a, b]$, prove that $f.g$ is also continuous at $x_0$.

(b) Define Supremum of a bounded set. Show that Supremum of a set $S \subseteq R$ whenever it exists is unique. 4

(c) Show that a real number p is a limit point of a set $S \subseteq R$ if and only if every neighbourhood of $p$ contains infinitely many points of S. $5\frac{1}{2}$

2. (a) Show that a finite set has no limit points. 6

(b) State and prove Bolzano-Weierstrass theorem. $6\frac{1}{2}$

### Section II

3. (a) Let $<a_n>$ be a sequence such that $a_n \neq 0$ for any $n$ and $\lim\limits_{x \to 0} \frac{a_{n+1}}{a_n} = l$.

Show that $a_n \to 0$ provided $|l| < 1$. 8

(b) Show that the sequence $<n^2>$ is not a Cauchy sequence. $4\frac{1}{2}$

4. (a) Let $<a_n>$ be a sequence such that

$$a_1 = \frac{3}{2} \text{ and } a_{n+1} = 2 - \frac{1}{a_n} \ \forall n.$$

Show that the sequence converges and find its limit. $6\frac{1}{2}$

(b) If $<a_n>$ and $<b_n>$ are bounded sequences such that $a_n \leq b_n \ \forall$ n, show that $\lim\sup a_n \leq \lim\inf b_n$. 6

### Section III

5. (a) Let $\sum\limits_{1}^{\infty} u_n$ be a positive term series such that $\lim\limits_{n \to \infty} u_n^{1/n} = 1$. Show that the series converges if $l < 1$, diverges if $l > 1$ and for $l = 1$, the series may converge or diverge. 7

(b) Show that the series

$$\frac{1}{1^p} + \frac{2}{2^p} + \frac{3}{3^p} + \ldots + \frac{n}{n^p} + \ldots$$

converges if $p > 2$ and diverges if $p \leq 2$. $5\frac{1}{2}$

6. (a) Define absolute convergence and show that an absolutely convergent series is also convergent. $5\frac{1}{2}$

(b) Show that the series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

converges absolutely for all values of $x$. 4

(c) Test for absolute convergence the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n+1!}.$$ 3

## Hons./I—1994

### MATHEMATICS—Unit IV (Analysis–II)

Time : 2 hours *Maximum Marks : 50*

*Attempt any two parts from each Section.*

### Section I

1. (a) Let $f$ be function defined on $[-1, 1]$ by $f(x) = x$ if $x$ is irrational, $f(x) = 0$ if $x$ is rational. Show that $f$ is continuous only at $x = 0$. $6\frac{1}{2}$

(b) Let $f$ be continuous on $[0, 1]$ and let $f(x)$ be in $[0, 1]$ for each $x$ in $[0, 1]$. Prove that $f(x) = x$ for some $x$ in $[0, 1]$. $6\frac{1}{2}$

(c) Show that every continuous function on a closed and bounded interval is uniformly continuous. $6\frac{1}{2}$

### Section II

2. (a) Let f be defined by setting—

$$f(x) = |x - 1| + |x + 1|, \text{ for all } x \in \mathbf{R}.$$

Show that f is not derivable at the points $x = -1$ and $x = 1$, and is derivable at every other point. $6\frac{1}{2}$

(b) (i) Let $f$ be derivable at $x_0$ and let $f(x_0) \neq 0$. Show that the function $1/f$ is derivable at $x_0$.

(ii) State the Inverse Function Theorem for derivatives. $4 + 2\frac{1}{2}$

(c) If $f'$ is continuous on $[a, a+h]$ and derivable on $]a, a+h[$, then prove that there exists a real number $c$ between $a$ and $a + h$ such that—

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(c). \qquad 6\frac{1}{2}$$

### Section III

3. (a) Evaluate :

(i) $\lim\limits_{x \to \pi/2} \dfrac{\tan 3x}{\tan x}$, (ii) $\lim\limits_{x \to (\pi/2) - 0} (\pi/2 - x)^{\tan x}$

(b) Show that the function $f$ defined by

$$f(x) = x^p (1 - x)^q, \text{ for all } x \in \mathbf{R},$$

where $p$ and $q$ are positive integers, has a maximum value, whatever the values of $p$ and $q$ may be.

(c) Show that $\sin x$ lies between—

$$x - \frac{x^3}{6} \text{ and } x - \frac{x^3}{6} + \frac{x^3}{120}.$$

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## B.Sc. (G)/III – 1995

### MATHEMATICS — Paper V

### (Real Analysis)

hours *Maximum Marks : 75*

*Attempt any* **two** *parts from each question.*

***All questions are compulsory and carry equal marks.***

) For any real number $x$, $y$; show that

(i) $|x + y| \leq |x| + |y|$

(ii) $\big||x| - |y|\big| \leq |x - y|$

(iii) $|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$.

Define a neighbourhood of a point. Show that the intersection of the family of all neighbourhoods of a point $x$ is $\{x\}$.

Define a closed set. Prove that the intersection of an arbitrary family of closed sets is a closed set. Is the conclusion true for the union of an arbitrary family of closed sets ? Justify.

) If $\lim\limits_{n \to \infty} a_n = l$ and $S_n = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$, show that

$$\lim_{n \to \infty} S_n = l.$$

(b) Define a Cauchy sequence. State (without proof) the Cauchy's general principle of convergence of a sequence. Show that the sequence $< a_n >$, where

$$a_n = 1 + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3n - 2}, \text{ does not converge.}$$

(c) Show that a real number L is the limit inferior of a bounded sequence $< a_n >$ if and only if —

(i) for each $\in > 0$, $a_n < L + \in$ for infinitely many values of $n$, and

(ii) for each $\in > 0$, $a_n > L - \in$ for all except finitely many values of $n$.

(a) Discus the convergence of the series whose $n$th term is

(i) $\sqrt[3]{n^3 + 1} - n$. (ii) $\dfrac{n^{n^2}}{(n + 1)^{n^2}}$

## B.A./B.Sc. (Hons.) I – 1995

## MATHEMATICS — Unit III (Analysis-I)

*Time* : 2 hours *Maximum Marks* : 50

*Attempt Four questions in all,*
*selecting only one question from each Section.*

### Sectior I

1. (*a*) Define Supremum of a set bounded above and show that it is unique. 3

(*b*) Show that :
$|x+y|^2+|x-y|^2=2|x|^2+2|y|^2,\ \forall\, x, y \in \mathrm{R}.$ 4½

(*c*) Show that a real no. $k$ is the infimum of a set $\mathrm{A} \subseteq \mathrm{R}$ bounded below if and only if

(*i*) $x \geq k\ \forall\ x \in \mathrm{A}$

(*ii*) for each $\in\, > 0$, $\exists\, y \in \mathrm{A}$ such that $k \leq y < k + \in$. 5

2. (*a*) Define a closed set and limit point of a set. 2

(*b*) Show that arbitrary intersection of closed sets is closed. 5

(*c*) Show that a set is closed if and only if it contains the set of its limit points. 5½

### Section II

3. (*a*) If $\lim a_n = a$ and $a \neq 0$, show that there exists a positive no. $k$ and a positive integer $m$ such that $|a_n| > k$ whenever $n \geq m$. 5½

(*b*) Define a Cauchy sequence. Show that a sequence converges if and only if it is a Cauchy sequence. 7

4. (*a*) Let $< a_n >$ be a sequence defined as follows :

$a_1 = \dfrac{3}{2};\ a_{n+1} = 2 - \dfrac{1}{a_n},\ n > 1.$

Show that $< a_n >$ is monotonic and bounded and determine its limit. 6

6½

(*b*) If $< a_n >$ and $< b_n >$ are bounded sequences such that $a_n \leq b_n\ \forall\ n \in \mathrm{N}$, show that $\lim \operatorname{Sup} a_n \leq \lim \operatorname{Sup} b_n$. 6½

### Section III

5. (*a*) State and prove D' Alembert's Ratio Test.

**(b)** **Verify that :**

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}, 0 < u < v.$$

**Deduce that :**

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}.$$ 6½

**(c)** **If** $f(0) = 0$ **and** $f''(x)$ **exists on** $[0, \infty[$, **show that :**

$$f'(x) - \frac{f(x)}{x} = \frac{1}{2} x f''(c), 0 < c < x.$$

**Also deduce that if** $f''(x)$ **is positive for positive values of** $x$, **then** $f(x)/x$ **strictly increases in** $]0, \infty[$. 6½

## Section III

3. **(a)** **Obtain the Maclaurin's series expansion of** $\log(1 + x)$. 6

**(b)** **Find the values of** $p$ **and** $q$ **in order that :**

$$\lim_{x \to 0} \frac{x(1 - p\cos x) + q \sin x}{x^3} = \frac{1}{3}.$$ 6

**(c)** **Show that the height of a closed cylinder of given volume and least surface is equal to its diameter.** 6

4. (*a*) Prove that $\lim_{x \to c} f(x)$ exists and is equal to a number $l$ if and only if both left limit $\lim_{x \to c-0} f(x)$ and right limit $\lim_{x \to c+0} f(x)$ exist and are equal to $l$.

(*b*) Define continuity of a function at a point. Let $f$ be a function defined by :

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ isirrational.} \end{cases}$$

Show that $f$ is discontinuous at every point of R.

(*c*) Show that a continuous function on a closed bounded interval is bounded above. Give an example to show that the conclusion fails if the interval is not closed.

5. (*a*) State and prove Cauchy's Mean Value Theorem.

(*b*) Obtain Maclaurin's series expansion of :

$$f(x) = \log(1+x), \; -1 < x \leq 1.$$

(*c*) Let $f$ be a function which is continuous and derivable on $[a-h, a+h]$. Prove that there is a real number $\theta$ between 0 and 1 such that :

$$f(a+h) - 2f(a) + f(a-h) = h\,[f'(a+\theta h) - f'(a-\theta h)].$$

6. (*a*) Prove that : $x - \dfrac{x^2}{2} < \log(1+x) < x - \dfrac{x^3}{2(1+x)}$, for $x > 0$.

(*b*) Prove that the function $f(x) = 3|x| + 4|x-1|$, $x \in R$ has a minimum value equal to 3.

(*c*) Evaluate :

(*i*) $\lim_{x \to 0+0} (\cot x)^{\sin x}$

(*ii*) $\lim_{x \to 0} \left(\dfrac{1}{x} - \cot x\right)$.

## B.A. / B.Sc. (Hons.) / I–1996
## MATHEMATICS – UNIT III
### (Analysis-I)

*Time : 2 hours*

*Maximum Marks : 50*

*Attempt **one** question only from each Section*

### SECTION I

1. (*a*) Show that

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|} \; ; x, y \in R.$$

(c) Discuss the convergence of the series :

$$\sum_{n=1}^{\infty} \frac{\sin nx + \cos nx}{n^{3/2}}.$$

6. (a) Define an alternating series. State and prove Leibnitz test for its convergence.

(b) Test for convergence and absolute convergence the series.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(n+1)!}.$$

(c) Test the convergence of the series $\sum_{n=1}^{\infty} \cos\frac{1}{n^2}$.

## B.A. / B.Sc. (Hons.) / I – 1996
## MATHEMATICS – Unit IV
## (Analysis – II)

*Time : 2 hours* *Maximum Marks : 50*

*All questions are compulsory.*

*Attempt any **two** parts from each question.*

1. (a) Prove that if a function is continuous on a closed and bounded interval, then it is bounded above and attains its supremum in that interval.

(b) Show that the function $f$ defined by :

$$f(x) = \begin{cases} x, & \text{when } x \text{ is rational} \\ -x, & \text{when } x \text{ is irrational} \end{cases}$$

is continuous at $x = 0$ only.

(c) Show that every continuous function on a closed and bounded interval is uniformly continuous.

2. (a) State and prove Cauchy mean value theorem. Using it show that :

$\sin\alpha - \sin\beta = \cot\theta\,(\cos\beta - \cos\alpha)$, where $0 < \alpha < \theta < \beta < \pi/2$.

(b) Examine the function :

$$f(x) = \begin{cases} x^m \sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

for derivability at origin. Also determine the value of $m$ for which $f'(x)$ is continuous at origin.

(c) Prove that if a function is derivable at a point, then it is continuous at that point. By considering the function $f$ defined by :

$$f(x) = \begin{cases} x\,[1 + \frac{1}{3}\sin(\log x^2)], & x \neq 0 \\ 0, & x = 0 \end{cases}$$

show that converse is not true.

**3.** (*a*) Show that the function *f*, defined by $f(x) = \sin x\,(1 + \cos x)$, for all real $x$, has a maximum value when $x = \pi/3$.

(*b*) Evaluate :

(*i*) $\lim_{x \to 0+0} (\cot x)^{\sin x}$, (*ii*) $\lim_{x \to 1-0} \frac{\log(1-x)}{\cot(\pi x)}$.

(*c*) Prove that $0 < [\log(1+x)]^{-1} - x^{-1} < 1$, for all $x > 0$.

### B.A. (Pass) / III – 1996
### MATHEMATICS – Paper III
### (Statistics and Real Analysis)

*Time : 3 Hours* *Maximum Marks : 100*

*All* sections are compulsory and have equal marks.

Attempt any *two* parts from each Section.

## SECTION I

**1.** (*a*) Define neighbourhood of a point, limit point of a set. Prove that a point *p* is a limit point of a set S of real numbers if and only if every neighbourhood of *p* contains infinitely many points of S.

(*b*) Prove that intersection of two open sets is open. Is the result true for arbitrary intersection of open sets ? Justify your answer.

(*c*) Let *f* be continuous on $[a, b]$ and *c* be any real number between $f(a)$ and $f(b)$. Prove that there exists a real number $x_0 \in (a, b)$ such that $f(x_0) = c$.

(*d*) State Bolzano-Weierstrass theorem (without proof). Show by an example that the condition of boundedness cannot be relaxed.

## SECTION II

**2.** (*a*) Prove that : $\lim_{n \to \infty} \sqrt[n]{n} = 1$.

(*b*) State and prove Cauchy's principle of convergence.

(*c*) State and prove D'Alembert's ratio test for series of positive terms.

(*d*) Test for convergence the series :

(*i*) $\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} + \ldots$, (*ii*) $\sum_{n=1}^{\infty} \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} \frac{x^{2n+1}}{(2n+1)}$.

## SECTION III

**3.** (*a*) State and prove Rolle's theorem.

(*b*) Find the maxima and minima of the function defined by

$$f(x) = 4x^5 - 25x^4 + 40x^3 + 4.$$

(*c*) Obtain Maclaurin's series expansion of $f(x) = \sin 2x$.

(*d*) Evaluate : $\lim_{x \to \pi/2} (\sin x)^{\tan^2 x}$

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## DELHI UNIVERSITY EXAMINATION PAPERS
### B.Sc. (G) / III – 1997
### MATHEMATICS – Paper V (Real Analysis)

*Time : 3 hours* *Maximum Marks : 75*

***All questions are compulsory and carry equal marks.***

***Attempt any two parts from each question.***

## Section I

**1.** (*a*) Explain what is meant by the field of real numbers being order complete. Show that the field of rational numbers is not order complete.

(*b*) If $A'$ denotes the derived set of $A$, show that :

$$(A \cap B)' \subset A' \cap B'.$$

Does the reverse inclusion hold ? Justify your answer. Also show that the derived set of a bounded set is itself bounded.

(*c*) Given two nonempty subsets $A$ and $B$ of $R$, define the set $C$ by

$$C = \{x + y \mid x \in A, y \in B\}.$$

If each of $A$ and $B$ has a suprenum, show that $C$ has a supremum and :

$$\text{Sup } C = \text{Sup } A + \text{Sup } B.$$

## Section II

**2.** (*a*) Define limit superior and limit inferior of a sequence. Prove that if a sequence $\langle a_n \rangle$ converges to $l$, then :

$$\lim_{n \to \infty} \sup a_n = \lim_{n \to \infty} \inf a_n = l.$$

(*b*) If : $\lim_{n \to \infty} a_n = l$ and $S_n = \frac{a_1 + a_2 + \ldots + a_n}{n}$

show that : $\lim_{n \to \infty} S_n = l$.

(*c*) Define a Cauchy sequence. State Cauchy's general principle of convergence of a sequence. Show that the sequence $\langle a_n \rangle$, where

$$a_n = 1 + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3n-2}$$

does not converge.

## Section III

**3.** (*a*) State and prove Cauchy's *n*th root test.

(*b*) Test for convergence the following series :

(*i*) $\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \ldots \frac{(n+1)!}{3^n} + \ldots$ (*ii*) $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n + 1}$.

(c) Define absolute convergence of a series. Show that an absolutely convergent series is convergent. Is the converse true ? Justify your answer.

### Section IV

4. (a) Let $f$ be a continuous function on $[a, b]$ and $c$ be any real number between $f(a)$ and $f(b)$. Show that there exists a real number $x$ in $]a, b[$ such that $f(x) = c$.

(b) Define uniform continuity of a function $f$ defined on an interval. Discuss the uniform continuity of the function :

$$f(x) = \sin x^2 \text{ on } [0, \infty[.$$

(c) Examine for continuity at $x = 0$, the function $f$ defined as :

$$f(x) = \begin{cases} \dfrac{x + |x|}{|x|} & , \quad x \neq 0 \\ 2 & , \quad x = 0. \end{cases}$$

### Section V

5. (a) State and prove Cauchy's Mean Value theorem.

(b) Obtain Maclaurin's series expansion of : $f(x) = \sin x, x \in \mathbf{R}$.

(c) Show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x = 1$

### Section VI

6. (a) Evaluate :

(i) $\lim\limits_{x \to 0} \left(\cot^2 x - \dfrac{1}{x^2}\right)$, (ii) $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$.

(b) Prove that : $\dfrac{x}{1+x} < \log(1+x) < x$ for $x > -1, x \neq 0$.

(c) Show that the function $f$ defined by :

$$f(x) = \sin x\,(1 + \cos x) \quad \forall\ x \in \mathbf{R}$$

has a maximum value when $x = \pi/3$.

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## B.A./B.Sc. (Hons.) I – 1997
## MATHEMATICS–Unit III
## (Analysis–I)

*Time : 2 Hours* *Maximum Marks : 50*

*Attempt Four questions in all, selecting only one question from each Section.*

### Section I

1. (a) Show $|a - b| \leq c$ *iff* $b - c \leq a \leq b + c \ \forall\ a, b, c \in \mathbf{R}$.

(b) Let $S$ and $T$ be non-empty sub-sets of $\mathbf{R}$ with the following property :

$$s \leq t \ \forall\ s \in S \text{ and } t \in T.$$

(i) Show that $S$ is bounded above and that $T$ is bounded below.

(ii) Show that Sup $S \leq$ Inf $T$.

(c) Let $S$ be the set :

$$\left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\ldots (-1)^n \frac{1}{n}, \ldots\right\}.$$

Find Sup $S$ and Inf $S$.

2. (a) If $M$ is a neighbourhood of a point $p$ and $N \supseteq M$, then $N$ is also a neighbourhood of $p$.

(b) Show that the union of an arbitrary family of open sets is an open set. Is every infinite set open ? Justify your answer by a proof or a counter-example.

(c) Can a non-empty finite set be a neighbourhood of any of its points ? Justify your answer.

3. (a) Show that a monotone increasing sequence that is bounded above converges.

(b) Show that : $\lim\limits_{n \to \infty} n^{1/n} = 1$.

4. (a) Let $r = 1$ and $s_{n+1} = \dfrac{(s_n + 1)}{3} \ \forall\ n \geq 1$

(i) Find $s_2, s_3, s_4$. (ii) Use induction to show that $s_n > \frac{1}{2} \ \forall\ n$.

(iii) Show that $< s_n >$ is a non-decreasing sequence.

(iv) Show that lim $s_n$ exists and find lim $s_n$.

(b) Let $< a_n >$ and $< b_n >$ be bounded sequence.

Show that lim Sup $(a_n + b_n) \leq$ lim Sup $s_n$ + lim Sup $b_n$.

### Section III

5. (a) State and prove D'Alembert's Ratio Test.

(b) (i) Show that the series $\Sigma\, a_n$, where :

$$a_n = 2^{(-1)^n} - n \text{ converges.}$$

(ii) Test the series for convergence :

$$\sum_{n=1}^{\infty} \frac{(a+x)^n}{n}, (a > 0).$$

6. (a) Show that an absolutely convergent series is always covergent. Give an example to show that the converse is not always true.

(b) Test for convergence and absolute convergence the series :

(i) $\Sigma\,(-i)^{n-1} \dfrac{n^2}{(n+1)\,!}$; (ii) $\sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n^2}$.

## B.A. (Pass) – 1997
## MATHEMATICS–Paper III (*ii*)
## (Statistics and Real Analysis)

*Time : 3 Hours* *Maximum Marks : 100*

***Attempt any Two parts from each Section.***

***All sections are compulsory. All questions carry equal marks.***

### Section I

1. (*a*) Define neighbourhood of a point. Show that intersection of two neighbourhoods of a point is also a neighbourhood of it.

(*b*) Show that every function defined and continuous on a closed bounded interval is bounded above therein.

(*c*) If $f$ be a function defined on $[-1, 1]$ by

$$\begin{aligned} f(x) &= x \quad , \quad \text{if } x \text{ is irrational} \\ &= 0 \quad , \quad \text{if } x \text{ is rational,} \end{aligned}$$

then show that $f$ is continuous only at $x = 0$.

(*d*) Show that $\sin x$ is uniformly continuous on $[0, \infty[$.

### Section II

2. (*a*) If a sequence $< a_n >$ in $R$ converges to $l \in R$, prove that the sequence :

$$< \frac{a_1 + a_2 + \ldots + a_n}{n} > \text{ also converges to } l.$$

(*b*) Prove that if : $a_n = \frac{1}{n} \{ (n+1)(n+2) \ldots (n+n) \}^{1/n}$

then $< a_n >$ converges to $4/e$.

(*c*) State and prove Raabe's Test for positive term series.

(*d*) Test the series : $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots$

for convergence for all positive values of $x$.

### Section III

3. (*a*) State and prove Cauchy's mean value theorem.

(*b*) Obtain Maclaurin's series expansion of

$$f(x) = \log(1 + x), \; -1 < x \leq 1.$$

(*c*) Obtain the points of maxima and minima for the function :

$$f(x) = (x-3)^2 (x+3)$$

(*d*) (*i*) $\lim\limits_{x \to 0} \left( \frac{1}{x} - \cot x \right)$;

(*ii*) $\lim\limits_{x \to 0+} (\cot x)^{\sin x}$

## DELHI UNIVERSITY EXAMINATION PAPERS
### B.Sc. (G) III – 1998
### MATHEMATICS – Paper V
### (Real Analysis)

*Time : 3 hours* *Maximum Marks : 75*

*All questions are compulsory and carry equal marks.*
*Attempt any two parts from each question.*

**1.** (*a*) Define the terms (*i*) bounded below, (*ii*) infimum of a set of real numbers.

Show that a real number $K$ is the infimum of a set $A \subseteq R$ iff

(*i*) $x \geq K \quad \forall \; x \in A$

(*ii*) for each $\varepsilon > 0$, $\exists \; y \in A$ such that $K \leq y < K + \varepsilon$.

(*b*) Define a closed set and limit point of a set. Show that a set is closed if and only if it contains all its limit points.

(*c*) Let $S$ be a nonempty closed and bounded subset of real numbers. Show that $S$ has a minimum and a maximum.

**2.** (*a*) Define limit superior of a sequence. If $\langle a_n \rangle$ and $\langle b_n \rangle$ are two bounded sequences, then show that :

$$\lim_{n \to \infty} \text{Sup}\,(a_n + b_n) \leq \lim_{n \to \infty} \text{Sup}\, a_n + \lim_{n \to \infty} \text{Sub}\, b_n .$$

(*b*) Let $\langle a_n \rangle$ be a sequence such that $a_n \neq 0$ for any $n$ and –

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l.$$

Prove that if $|l| < 1$, then $\lim_{n \to \infty} a_n = 0$.

(*c*) Prove that a sequence $\langle a_n \rangle$ defined by –

$$a_1 = 0$$

$$a_n = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \cdots \frac{1}{(n-1)\,!}, \quad n \geq 2$$

is convergent.

**3.** (*a*) State and prove Raabe's Test for the convergence of a positive term series.

(*b*) Test for convergence the following series :

(*i*) $\sum_{n=1}^{\infty} (\sqrt{n^4+1} - \sqrt{n^4-1})$, (*ii*) $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$.

(*c*) If an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$, $u_n > 0$ satisfies :

i) $u_{n+1} \leq u_n \;\; \forall \; n$, (*ii*) $\lim_{n \to \infty} u_n = 0$,

then show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ converges.

Let $f$ be a continuous function on $[a, b]$ and let $M = \text{Sup}\, f$, $m = \text{Inf}\, f$. Show that $f$ attains every value between $m$ and $M$.

Define uniform continuity of a function. Show that every uniformly continuous function is continuous. Give an example of a continuous function which is not uniformly continuous.

) Let $f$ be a function defined on $[0, 1]$ by –

$$f(x) = \frac{1}{2^n}, \quad \text{when } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}, \; n = 0, 1, 2, \ldots$$

$$f(0) = 0.$$

Examine $f$ for discontinuities.

State and prove Rolle's Theorem.

Obtain Maclaurin's series expansion of $f(x) = e^x, x \in R$.

) Show that there exists $\theta$ satisfying :

$$\frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \theta, \text{ where } 0 < \alpha < \theta < \beta < \frac{\pi}{2}.$$

) Evaluate :

(*i*) $\lim_{x \to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x}$, (*ii*) $\lim_{x \to 1} (1 - x) \tan\left(\frac{\pi}{2} x\right)$

) Prove that : $|\tan^{-1} x - \tan^{-1} y| < |x - y| \quad \forall \; x, y \in R$.

) Given the perimeter of a rectangle, show that its area is maximum when it is a square.

### B.A./B.Sc. (Hons.) / I – 1998
### MATHEMATICS – Unit III
### (Analysis-I)

*Maximum Marks : 50*

*: 2 Hours*

*Attempt Four questions in all, selecting only one question from each Section*

### Section I

✓) For all real numbers $x$ and $y$, show that :

$$|x + y| \leq |x| + |y|.$$

Hence, or otherwise, prove that :

$$\sqrt{|x+y|} \leq \sqrt{|x|} + \sqrt{|y|}.$$

(b) Test for convergence the series :

(i) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p},\ p>0$ ;

(ii) $\sum_{n=2}^{\infty} u_n = 1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\frac{30}{33}x^4+\ldots$

(a) State Cauchy's integral test for the convergence of an infinite series of positive terms. Use it to prove that $\Sigma \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \le 1$.

(b) (i) Test the convergence of the following series :

$$\sum_{n=1}^{\infty} \frac{3.6.9\ldots.3n}{7.10.13\ldots.(3n+4)} x^n,\ x>0.$$

(ii) Test the following series for convergence and absolute convergence :

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}.$$

### B.A./B.Sc. (Hons.) /I – 1998
### MATHEMATICS–Unit IV
### (Analysis-II)

Time : 2 Hours

Maximum Marks : 50

*All questions are compulsory*
*Attempt any two parts from each question.*

1. (a) If the function $f$ defined on an interval $I$ is continuous at the point $p \in I$, then show that $|f|$ is continuous at $p \in I$. Is the converse true ? Give arguments in support of your answer.

(b) Examine the continuity of the function :

$$f(x)=\begin{cases} \dfrac{e^{1/x^2}}{1-e^{1/x^2}}, & \text{when } x \ne 0 \\ 0, & \text{when } x=0 \end{cases}$$

at $x=0$.

(c) Show that every uniformly continuous function is continuous. Give an example of a continuous function which is not uniformly continuous.

2. (a) Prove that if $P$ be any polynomial and $P'$ the derivative of $P$, then between any two consecutive zeros of $P'$, there lies at the most one zero of $P$.

(*b*) If $f'(x)$ and $g'(x)$ exist for all $x$ in $[a, b]$, and if $g'(x)$ does not vanish anywhere in $]a, b[$, then prove that for some $c$ between $a$ and $b$ :

$$\frac{f(c)-f(a)}{g(b)-g(c)}=\frac{f'(c)}{g'(c)}.$$

(*c*) Let $f$ be the function on **R** defined by

$$f(x)=|x-2|+|x+2|,$$

for all $x \in \mathbf{R}$. Show that $f$ is not derivable at the points $x=-2$ and $x=2$, and is derivable at every other point.

3. (*a*) (*i*) Obtain the Maclaurin's series expansion of

$$(1+x)^m, \quad \text{where} \quad m<1.$$

(*ii*) Evaluate : $\lim\limits_{x \to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$.

(*b*) If $f''(x)>0$ for all $x \in \mathbf{R}$, then show that

$$f\left\{\frac{1}{2}(x_1+x_2)\right\} \leq \frac{1}{2}[f(x_1)+f(x_2)]$$

for every pair of real numbers $x_1$ and $x_2$.

(*c*) Show that the function $f$, defined by $f(x)=|x|^m|x-1|^n$, for all $x \in \mathbf{R}$, has a maximum value

$$\frac{m^m n^n}{(m+n)^{m+n}}, \; m \text{ and } n \text{ being positive.}$$

# B.A.(P) / III – 1998

## MATHEMATICS – Paper III

## (Statistics and Real Analysis)

*Time : 3 Hours* *Maximum Marks : 100*

*All Sections are compulsory and have equal marks.*
*Attempt any two parts from each Section.*

### Section 1

1. (*a*) If $S$ be a non-empty set of real numbers which is bounded above then show that a real number $s$ is the supremum of $S$ iff the following two conditions hold :

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$x \leq s \;\forall\; x \in S$ ;

For each positive real number $\varepsilon$, there is a real number $x \in S$ such that $x > s - \varepsilon$.

efine an open set and show that intersection of two open sets in pen.

how that the function $f$ defined by :

$$f(x)=\begin{cases}+1, & x \text{ is rational}\\ -1, & x \text{ is irrational}\end{cases}$$

s discontinuous at every point of **R**.

### Section 2

Define a convergent sequence and a bounded sequence. Show that every convergent sequence is bounded.

Show that the sequence $\langle a_n \rangle$ defined by the reaction :

$$a_1=1,\; a_n=1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{(n-1)!},\; (n \geq 2) \text{ converges.}$$

State and prove Cauchy's $n$th Root Test for the convergence of a positive term series.

Test the convergence of the series :

$$\sum_{n=1}^{\infty}\frac{1\,.\,3\,.\,5\,.\,\ldots\,(4n-5)\,(4n-3)}{2\,.\,4\,.\,6\,.\,\ldots\,(4n-4)\,(4n-2)}\,\frac{x^{2n}}{4n}.$$

### Section 3

Obtain the Maclaurin's series expansion of $\sin x$.

Applying Lagrange's mean value theorem to the function defined by

$$f(x)=\log(1+x) \;\forall\; x \geq 0,$$

show that : $0<[\log(1+x)]^{-1}-x^{-1}<1$ whenever $x>0$.

Evaluate :

(*i*) $\lim\limits_{x \to 0}\frac{\tan x - x}{x^2 \tan x}$ ;

(*ii*) $\lim\limits_{x \to 0^+}(\cot x)^{\sin x}$.

## B.Sc. (G)/III – 1999
## MATHEMATICS – Paper V
## (Real Analysis)

*Maximum Marks* : 75

*Time* : *3 hours*

*Attempt any* **two** *parts from each question.*

***All*** *questions are compulsory and carry equal marks.*

1. (*a*) State the properties which describe the set of all real numbers as a complete ordered field.
Show that necessary conditions for a number $\alpha$ to be the supremum of a non-empty set S are :
(*i*) $x \le \alpha \ \forall x \in S$
(*ii*) For each $\varepsilon > 0$, $\exists y \in S$ such that $y > \alpha - \varepsilon$.

(*b*) Define (*i*) open set, (*ii*) closed set.
Prove that the set $\left\{-1, 1, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{3}, \frac{1}{3}, \ldots\right\}$ is neither open nor closed.

(*c*) Let $S$ be a bounded infinite non-empty subset of **R**. Prove that the derived set of $S$ has the smallest and the greatest members.

2. (*a*) Define limit superior and limit inferior of a sequence.
Let $(a_n)$ be a real bounded sequence and $(b_n)$ a real convergent sequence. Show that :
$$\lim \operatorname{Sup} (a_n + b_n) = \lim \operatorname{Sup} a_n + \lim b_n.$$

(*b*) If $\lim\limits_{n \to \infty} a_n = 0$ and $a \neq 0$, show that there exists a positive number $k$ and a positive integer $m$ such that $|a_n| > k$ whenever $n \ge m$.

(*c*) Let $(a_n)$ be a sequence of positive numbers such that
$$a_0 = 0,\ a_1 = 1,\ a_n = \frac{1}{2}(a_{n-1} + a_{n-2})\ \forall\ n \ge 2.$$
Show that $(a_n)$ converges and find its limit.

3. (*a*) State and prove Cauchy's $n$th root test.

(*b*) Discuss the convergence of the series :
(*i*) $x^2 (\log 2)^p + x^3 (\log 3)^p + x^4 (\log 4)^p + \ldots\ (x > 0)$
(*ii*) $\sum\limits_{n=1}^{\infty} u_n$, where $u_n = \cos \frac{1}{n^2}$.

(*c*) Test for convergence and absolute convergence the series :
$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots.$$

4. (*a*) **Let $f$ be continuous on** $[a, b]$. Prove that $f$ attains its supremum on $[a, b]$.

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closed and finite interval but is not uniformly continuous on **R**.

(*a*) State Intermediate value theorem for a continuous function defined on $[a, b]$.
Let $f$ be continuous on $[a, b]$ and $x_1, x_2, \ldots, x_n$ be points of $[a, b]$. Show that there exists a point $c \in [a, b]$ such that :
$$f(c) = \frac{1}{n}[f(x_1) + f(x_2) + \ldots + f(x_n)].$$

(*a*) State and prove Cauchy Mean value theorem.

(*b*) Obtain the Maclaurin's series expansion of $f(x) = (1 + x)^m$.

(*c*) Show that there exists no real number $k$ such that the equation $x^3 - 3x + k = 0$ has two distinct roots in $[0, 1]$.

(*a*) Prove that : $1 - \frac{x^2}{2\,!} \le \cos x \le 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!}$, $\forall\ x \in \mathbf{R}$.

(*b*) Examine for extreme values the function $f(x)$ defined by—
$$f(x) = |x|^m f(x) = |x - 1|^n,\ \forall\ x \in \mathbf{R}$$
$m$ and $n$ being positive integers.
Also show that it has a maximum value $\frac{m^m \cdot n^n}{(m+n)^{m+n}}$.

(*c*) Find the values of $a$ and $b$ such that
$\lim\limits_{x \to 0} \frac{x(1 - a \cos x) + b \sin x}{x^3}$ exists and is equal to $\frac{1}{3}$.

## B.A./B.Sc. (Hons.)/I – 1999
## MATHEMATICS – Unit III
## (Analysis — I)

*Maximum Marks* : 50

*Time* : *2 hours*

*Attempt* ***Four*** *question in all, selecting only* ***one question from each Section.***

### Section I

1. (*a*) (*i*) Define neighbourhood of a point in **R.** Show that the intersection of the family of all neighbourhoods of an arbitrary point $x \in R$ is $\{x\}$.
(*ii*) Is every infinite set open ? Justify your answer.

(*b*) Let $S$ and $T$ be non-empty subsets of **R** with the following property :
$$s \le t,\ \forall\ s \in S, t \in T.$$
(*i*) Show that $S$ is bounded above and that $T$ is bounded below.
(*ii*) Show that : sup S ≤ inf T.

(*c*) Let $S = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots, (-1)^n \frac{1}{n}, \ldots\right\}$, find sup S and inf S.

**2.** (*a*) (*i*) Can a non-empty finite set be a neighbourhood of any of its points ? Justify your answer.

(*ii*) Prove that a non-empty bounded closed set contains its supremum as well as its infimum.

(*iii*) For a non-empty bounded subset $S$ of **R**, prove that :

$$\inf S \leq \sup S.$$

(*b*) (*i*) If $x$ is a limit point of a set $S \subset \mathbf{R}$, then is $x$ a limit point of $S - \{x\}$ also ? Justify your answer.

(*ii*) Prove that : $\sup (S \cup T) = \max \{ \sup S, \sup T\}$, where S and T are arbitrary non-empty bounded subsets of **R**.

(*iii*) Show that : $\sup \{r \in \mathbf{Q} : r < a\} = a$, for each $a \in \mathbf{R}$.

## SECTION II

**3.** (*a*) (*i*) Let $< a_n >$ be a convergent sequence of real numbers such that $a_n \neq 0$ for all $n \in \mathrm{N}$ and $\lim < a_n > = a \neq 0$.

Prove that : $\inf \{| a_n | : n \in \mathrm{N} \} > 0$.

(*ii*) Define subsequence of a sequence $< a_n >$ and prove that $p$ is a limit point of a sequence $< a_n >$ if and only if there exists a subsequence of $< a_n >$ converging to $p$.

(*a*) (*i*) Prove that every bounded sequence has a limit point and the set of limit points of a sequence is bounded.

(*ii*) Give an example of an unbounded sequence with exactly two limit points.

**4.** (*a*) (*i*) Prove that a bounded sequence with unique limit point is convergent. Can the conditions in the above theorem be relaxed ?

(*ii*) Let $s_1 = \sqrt{2}$ and let $s_{n+1} = \sqrt{2}\,\sqrt{s_n}$ for $n \geq 2$. Prove that :

$$\lim_{n \to \infty} < s_n > = 2.$$

(*b*) (*i*) If the limit inferior of a sequence $< a_n >$ is $m$, then prove that no subsequence of $< a_n >$ can converge to a limit less than $m$.

(*ii*) If $< a_n >_{n=1}^{\infty}$ is a sequence of real numbers which converges to $l$, then $< a_n^2 >_{n=1}^{\infty}$ converges to $l^2$.

## SECTION III

**5.** (*a*) (*i*) Define an infinite series. When is a series said to be oscillating ? Can a positive term series be oscillating ?

(*ii*) Prove that if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges.

(*i*) Prove that a necessary and sufficient condition for an infinite series of positive terms to be convergent is that the sequence of its partial sums is bounded above.

(*ii*) Test for convergence the following series :

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n\,!}},\ \sum_{n=1}^{\infty} (-1)^n\, n,\ \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{3}\right).$$

**5.** (*a*) Define an alternating series. State and prove Leibnitz's test for the convergence of an alternating series. Use the test to prove that the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}$ is convergent.

(*b*) Show that an absolutely convergent series is always convergent. Give examples to show that the converse may or may not be true.

(*c*) Test for convergence the series : $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

### B.A./B.Sc. (Hons.)/I – 1999

### MATHEMATICS – Unit IV

### (Analysis—II)

*Time : 3 hours* *Maximum Marks : 50*

*All questions are compulsory. Attempt any two parts from each question.*

**1.** (*a*) Show that a real-valued function $f$ defined on an interval $I$ is continuous at $p \in I$ if and only if for every sequence $< p_n >$ in $I$ which converges to $p$, we have $\lim_{n \to \infty} f(p_n) = f(p)$.

(*b*) Let $f$ be defined on R as $f(x) = \begin{cases} 1, & \text{when } x \text{ is irrational} \\ -1, & \text{when } x \text{ is rational.} \end{cases}$

Show that $f$ is discontinuous at every point of **R**.

(*c*) Show that every real-valued continuous function on a closed and bounded interval is uniformly continuous.

**2.** (*a*) Let $f$ be defined on **R** as

$$f(x) = \begin{cases} x\left(\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right) & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0. \end{cases}$$

Show that $f$ is not derivable at $x = 0$.

(*b*) If $f$ and $g$ defined on an interval $I$ are derivable at $x_0 \in I$, then show that $fg$ is also derivable at $x_0$.

(*c*) If $f'$ is continuous on $[a, a+h]$ and derivable on $]a, a+h[$, then prove that there exists a real number $c$ between $a$ and $a+h$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(c).$$

**3.** (*a*) Evaluate : (*i*) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$; (*ii*) $\lim_{x \to \frac{\pi}{2} - 0} (\tan x)^{\sin 2x}$.

# DELHI UNIVERSITY EXAMINATION PAPERS
## B.Sc. (G) /II–2000
## MATHEMATICS – Paper V
## (Real Analysis)

Time : 3 hours — Maximum Marks : 75

*All* questions are compulsory and carry equal marks.
Attempt any *two* parts from each question.

1. (*a*) Define an upper bound of a set $S$ and its supremum. Show that a real number $k$ is the supremum of a non-empty set $S$ if and only if the following two conditions hold :

   (*i*) $x \leq k$ for all $x \in S$.

   (*ii*) For each positive real number $\varepsilon > 0$, there is a real number $x \in S$ such that $x > k - \varepsilon$.

   (*b*) Define an open set. Show that the union of an arbitrary family of open sets is open but the intersection of an arbitrary family of open sets may fail to be open.

   (*c*) Define a limit point of a set. Show that the set $Z$ of integers has no limit point and every point of $[0, 1]$ is a limit point of the open interval $]0, 1[$

2. (*a*) Define a Cauchy sequence. Show that every convergent sequence is a Cauchy sequence and every Cauchy sequence is bounded.

   (*b*) Show that the sequence $\{r^n\}$ converges to zero if $|r| < 1$.

   (*c*) Define a monotonic sequence. Let $\langle a_n \rangle$ be a sequence defined as follows :

$$a_1 = \frac{3}{2} ; a_{n+1} = 2 - \frac{1}{a_n}, n \geq 1.$$

   Show that $\langle a_n \rangle$ is monotonic and bounded. Find the limit of this sequence.

3. (*a*) State and prove D'Alembert's ratio test for positive term series.

   (*b*) Test for convergence the following series :

   (*i*) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$

   (*ii*) $\sum_{n=1}^{\infty} (n^{1/n} + x)^n$ for all positive values of $x$.

   (*c*) Define absolutely convergent series. Show that every absolutely convergent series is convergent but a convergent series may not be absolutely convergent.

4. (*a*) Show that if a function $f$ is continuous on the closed and finite interval $[a, b]$, then it attains its supremum at some point of $[a, b]$.

(*b*) Find Sup $S$ and Inf $S$ for the set :

$$S = \left\{ (-1)^n \frac{1}{n} ; n \in N \right\}.$$

(*c*) What are the derived sets of :

(*i*) the set of integers ;

(*ii*) the set of rational numbers ;

(*iii*) A finite set.

Justify your answers.

**2.** (*a*) State and prove Archimedean property of real numbers. Deduce that for every real number $x$ there exists a positive integer $n$ such that $-n < x$.

(*b*) Prove that the intersection of an arbitrary family of closed sets is closed. What can you say about the intersection of an arbitrary family of open sets ? Justify your answer.

(*c*) Prove that a real number $k$ is the supremum of a set $A$ if and only if :

(*i*) $x \leq k \ \forall \ x \in A$ ;

(*ii*) For each $\varepsilon > 0$, $\exists \ y \in A$ such that $k - \varepsilon < y$.

## Section II

**3.** (*a*) Define a Cauchy sequence. Prove that a sequence of real numbers is convergent if and only if it is a Cauchy sequence.

(*b*) Show that the sequence $\{a_n\}$ given by

$$a_{n+1} = 2 - \frac{1}{a_n}, n \geq 1, n \in N$$

and $a_1 = \dfrac{3}{2}$

is convergent. Also find $\lim\limits_{n \to \infty} a_n$.

**4.** (*a*) State and prove the necessary and sufficient condition for a monotonic sequence to be convergent.

(*b*) Show that :

(*i*) for any real number $x$, $\lim\limits_{n \to \infty} \dfrac{x^n}{n!} = 0$.

(*ii*) $\lim\limits_{n \to \infty} \dfrac{m(m-1)(m-2)\ldots(m-n+1)}{n!} x^n = 0, |x| < 1$ ;

(*iii*) $\lim\limits_{n \to \infty} \dfrac{1}{n}\left[1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n}\right] = 1.$

Explain why the function $f$ does not vanish anywhere in $\left[0, \frac{\pi}{2}\right]$, although $f(0) \cdot f\left(\frac{\pi}{2}\right) < 0$.

(*b*) What do you understand by a uniformly continuous function ? Let $f(x)$ be a real valued function on the closed and bounded interval $I$. Show that $f$ is continuous on $I$ if and only if $f$ is uniformly continuous on $I$.

2. (*a*) (*i*) Let $f$ be derivable at $x_0$ and $f(x_0) \neq 0$. Show that the function $\frac{1}{f}$ is derivable at $x_0$ and $\left(\frac{1}{f}\right)'(x_0) = -\frac{f'(x_0)}{[f(x_0)]^2}$.

(*ii*) State the inverse function theorem for derivatives.

(*b*) Verify that :

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}, \quad \text{if} \quad 0 < u < v.$$

Deduce that :

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}.$$

(*c*) If $f'(x) > 0 \; \forall \; x \in R$, show that :

$$f\left[\frac{1}{2}(x+y)\right] \leq \frac{1}{2}[f(x) + f(y)] \; \forall \; x, y \in R.$$

3. (*a*) Assuming the validity of the expansion, show that :

$$\tan^{-1} x = \tan^{-1}\frac{\pi}{4} + \frac{x - \frac{\pi}{4}}{1 + \frac{\pi^2}{16}} - \frac{\pi\left(x - \frac{\pi}{4}\right)^2}{4\left(1 + \frac{\pi^2}{16}\right)} + \ldots$$

(*b*) Prove that the function $f$ defined by :

$$f(x) = |x-2| \cdot |x-3| \; \forall \; x \in R$$

has a minimum value 0 at 2, 3 and a maximum value $\frac{1}{4}$ at $\frac{5}{2}$.

(*c*) Evaluate :

(*i*) $\lim\limits_{x \to 0} \left[\frac{1}{e^x - 1} - \frac{1}{x}\right]$.

(*ii*) $\lim\limits_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$.

## B.A. (P) / III–2000

## MATHEMATICS – Paper III

## (Statistics and Real Analysis)

Time : 3 hours *Maximum Marks : 100*

*All* Sections are compulsory and have equal marks.
Attempt any *two* parts from each Section.

### Section I

(*a*) Define an open subset of **R**. Show that the intersection of finitely many open sets is an open set. Is the result true for intersection of any arbitrary family of open sets ? Justify your answer.

(*b*) If $S'$ denotes the set of limit points of $S$, then show that :

$$(S\cup T)' = S'\cup T' \quad\text{and}\quad (S\cap T)' \subsetneqq S'\cap T'.$$

(*c*) Let $f$ and $g$ be defined on $[a, b]$. If $f$ and $g$ are continuous at $x_0 \in [a, b]$ and $g(x_0) \neq 0$, show that $f/g$ is continuous at $x_0$.

(*d*) State and prove intermediate value theorem.

### Section II

(*a*) Show that $\lim\limits_{n\to\infty} n^{1/n} = 1$.

(*b*) If $\lim\limits_{n\to\infty} a_n = l$ and if $\langle s_n \rangle$ is a sequence defined as

$$s_n = \frac{a_1 + a_2 \ldots + a_n}{n}, \text{ then show that } \lim_{n\to\infty} s_n = l.$$

(*c*) State and prove Raabe's test for the convergence of series of positive terms.

(*d*) Test the convergence of

$$(i)\ \sum_{n=2}^{\infty} \frac{1}{n^2 \log n}\ ; \qquad (ii)\ \frac{x}{1.3} + \frac{x^2}{2.4} + \frac{x^3}{3.5} + \frac{x^4}{4.6} + \ldots$$

### Section III

) State and prove Cauchy's mean value theorem.

(*b*) Obtain the Maclaurin's series of $e^{4x}$.

(*c*) Evaluate :

$$(i)\ \lim_{x\to 0+}\left(\frac{\tan x}{x}\right)^{1/x^2}\ ; \qquad (ii)\ \lim_{x\to 0} \frac{a^x - 1 - x\log a}{x^2},\ a > 0.$$

) Find *two* numbers whose sum is 15 and the square of one multiplied by the cube of the other is maximum.

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## B.Sc. (G)/III–2001

## MATHEMATICS – Paper V

## (Real Analysis)

Time : 3 hours *Maximum Marks : 75*

*All questions are compulsory and carry equal marks.*
*Attempt any two parts from each question.*

1. (*a*) Define neighbourhood of a point, open set, closed set and limit point of a set. Show that a finite set has no limit point.

(*b*) Describe the properties which make the field of real numbers order complete. Show that the field of rational numbers is not order complete.

(*c*) Define a bounded set of real numbers and prove that an infinite bounded set of real numbers has a limit point.

2. (*a*) Define a convergent sequence and a bounded sequence. Show that every convergent sequence is bounded. Is the converse true ? Justify your answer.

(*b*) Define limit superior and limit inferior of a sequence $< a_n >$. Show that if a bounded sequence $< a_n >$ converges to a real number $l$, then

$$\lim\ \sup a_n = \lim\ \inf a_n = l.$$

(*c*) Define a monotonically increasing sequence defined by $a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2a_n}$ converges to 2.

3. (*a*) Let $\Sigma u_n$ be a series of positive terms such that

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = l. \text{ Show that :}$$

(*i*) if $l > 1$, the series is convergent ;

(*ii*) if $l < 1$, the series is divergent,

(*b*) Test for convergence the series :

$$(i)\ \frac{1^2 . 2^2}{1\,!} + \frac{2^2 . 3^2}{2\,!} + \frac{3^2 . 4^2}{3\,!} + \ldots. \qquad (ii)\ \sum_{n=1}^{\infty} \frac{n^{n^2}}{(n+1)^{n^2}}.$$

(*c*) Define absolutely convergent series. Show that the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n+2}{2^n+5} \cos(nx)$ is convergent for all real values of $x$.

4. (*a*) Show that a continuous function on a closed and finite interval $[a, b]$ is bounded on $[a, b]$.

(*b*) Examine the continuity of the function :

$$f(x) = \begin{cases} \dfrac{x + 2|x|}{|x|}, & \text{if } x \neq 0 \\ 3, & \text{if } x = 0 \end{cases}$$

on the interval $[0, 5]$.

## Section II

(a) Prove that $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$.

(b) State Cauchy's convergence criterion and show that the sequence $< a_n >$ defined by $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$ does not converge.

(c) State and prove D'Alembert's Ratio test for series of positive terms.

(d) Test the convergence of the following series :

(i) $\sum\limits_{n=1}^{\infty} (\sqrt{n^2+1} - n)$ ;

(ii) $\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \frac{x^4}{5\sqrt{6}} + \ldots. (x > 0)$.

## Section III

3. (a) State and prove Lagrange's mean value theorem.

(b) Obtain Maclaurin's series for $f(x) = \cos 3x$.

(c) Evaluate :

(i) $\lim\limits_{x \to 0+} (\cot x)^{\sin x}$ ;  (ii) $\lim\limits_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$.

(d) Examine the function $f(x)$ for maxima and minima where

$$f(x) = 2x^3 - 15x^2 + 36x + 10, x \in \mathbf{R}.$$

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# B.A./B.Sc. (Hons.)/I–2001

# MATHEMATICS – Unit III

## (Analysis — I)

*Time : 2 hours*  *Maximum Marks : 50*

*Attempt four questions in all, selecting only one question from each Section.*

## Section I

1. (a) For any real numbers $x$, $y$, prove that :

(i) $|x + y| \le |x| + |y|$.

(ii) $||x| - |y|| \le |x - y|$.

(iii) $|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$.

(b) State and prove Bolzano-Weierstrass Theorem for sets.

2. (a) Let $S$ be a non-empty bounded above set of real numbers. Prove that a real number $a$ is supremum of the set $S$ iff

(i) $x \le a, \ \forall \ x \in S$ ;

(ii) for each $\varepsilon > 0$, there exists $x_0 \in S$ such that $x_0 > a - \varepsilon$.

(*b*) Let $f$ be a function defined on R by

$$f(x) = \begin{cases} x, & \text{when } x \text{ is rational} \\ -x, & \text{when } x \text{ is irrational.} \end{cases}$$

Show that $f$ is continuous at $x = 0$ and discontinuous at any $c \neq 0$.

(*c*) Explain what you understand by a uniformly continuous function. Show that a uniformly continuous function on a closed, bounded interval I is continuous on I.

**2.** (*a*) Prove that if $P$ is any polynomial and $P'$ the derivative of $P$ then between any two consecutive zeros of $P'$, there lie at most one zero of $P$.

(*b*) If $f''(x) > 0 \ \forall \ x \in \mathbf{R}$, show that :

$$f\left[\frac{1}{2}(x+y)\right] \leq \frac{1}{2}[f(x) + f(y)] \text{ for all } x, y \in \mathbf{R}.$$

(*c*) Prove that if $f'(x)$ is continuous on $[a, a+h]$ and derivable on $]a, a+h[$, then prove that there exists a real number $c$ between $a$ and $a+h$ such that :

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2\,!} f''(c).$$

**3.** (*a*) Show that :

$$\frac{x}{1+x} < \log(1+x) < x, \quad \text{for } x > -1 \text{ and } x \neq 0.$$

(*b*) Obtain Maclaurin's series expansion of $e^x$.

(*c*) Show that $f(x) = |x|^m |x-1|^n$ for all $x \in \mathbf{R}$ has a maximum value

$$\frac{m^m n^n}{(m+n)^{m+n}},$$ $m$ and $n$ being positive integers.

## DELHI UNIVERSITY EXAMINATION PAPERS
### B.Sc.(G)/III – 2002
### MATHEMATICS – Paper V
### (Real Analysis)

*Time : 3 hours* *Maximum Marks : 75*

*All questions are compulsory and carry equal marks.*
*Attempt any two parts from each questions.*

**1.** (*a*) Define a lower bound and infimum of a set of real numbers. State the order completeness property of real numbers and apply it to prove that any non-empty set of real numbers which is bounded below has an infimum.

(*b*) Define a closed set and limit point of a set of real numbers. Show that a set of real numbers is closed if and only if it contains all its limit points and apply it to prove that the set $Z$ of integers is closed.

(*c*) Answer in 'yes' or 'no'. Justify your answer in each case by a proof or a counter example :

(*i*) Is the intersection of an arbitrary family of open sets, an open set ?

(*ii*) Is the union of an arbitrary family of open sets, an open set ?

(*iii*) Can a finite non-empty set be open ?

**2.** (*a*) Define a Cauchy sequence. State the Cauchy's general principle of convergence of a sequence. Show that the sequence $\langle a_n \rangle$, where

$$a_n = 1 + \frac{1}{4} + \frac{1}{7} + \ldots\ldots + \frac{1}{3n-2}$$

does not converge.

(*b*) Define limit superior and limit inferior of a bounded sequence $\langle a_n \rangle$. Show that if

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n = l,$$

then the sequence $\langle a_n \rangle$ converges to $l$.

(*c*) Let $\langle a_n \rangle$ be a sequence of positive real numbers such that : $\lim_{n\to\infty} a_n = a, a > 0$, then the sequence $\langle b_n \rangle$,

where $b_n = (a_1 a_2 \ldots\ldots a_n)^{1/n}$ converges to $a$.

**3.** (*a*) State and prove Raabe's test for the convergence of a positive term series.

(*b*) Test for convergence the following :

(*i*) $\sum_n \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$ ;

(*ii*) $\frac{2.4}{3.5} + \frac{2.4.6}{3.5.7}x + \frac{2.4.6.8}{3.5.7.9}x^2 + \ldots\ldots, x > 0.$

(*c*) Define a conditionally convergent and an absolutely convergent series. Show that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{2^n+5} \cos nx$$

is absolutely convergent.

**4.** (*a*) Let $f$ be a continuous function on $[a, b]$ and $x_0$ be such that $a < x_0 < b$, show that if $f(b) < 0$, then there exists a $\delta_0 > 0$ such that $f(x) < 0$ for all $x \in ]\,b - \delta_0, b\,]$.

(*b*) A function $f(x)$ is defined by

$$f(x) = \begin{cases} \dfrac{[x^2]-1}{x^2-1}, & \text{if } x^2 \neq 1 \\ 0, & \text{if } x^2 = 1, \end{cases}$$

where $[x]$ denotes the greatest integer function. Discuss the continuity of $f(x)$ at $x = 1$.

(*c*) Show that the function :

$$f(x) = \sin\frac{1}{x} \text{ for all } x \in ]\,0, \infty\,[$$

is not uniformly continuous on $]\,0, \infty\,[$.

**5.** (*a*) State and prove Rolle's theorem and apply it to show that the tangent to the graph of the function : $f(x) = \frac{1}{x^2+1}$ is parallel to $x$-axis, at least at one point between $-3$ and 3.

(*b*) Obtain Maclaurin's series expansion of the function $\sin x$.

(*c*) State Lagrange's Mean Value theorem and Cauchy's Mean Value theorem. Show that first is a particular case of the second. Give geometrical interpretation of Lagrange's theorem.

**6.** (*a*) For $x > 0$, show that :

$$\log\left(1 + \frac{1}{x}\right) < \frac{1}{\sqrt{x^2+n}}$$

(*b*) Suppose $f(x)$ is a function satisfying the following conditions :

(*i*) $f(0) = 2, f(1) = 1$ ;

(*ii*) $f$ has a minimum value at $x = 5/2$ ;

(*iii*) $f\,(x) = 2ax + b$ for all $x$.

Determine the constants $a$, $b$ and the function $f(x)$.

**4.** (*a*) **Show that a bounded sequence $\langle a_n \rangle \to l$ if and only if**

$$\lim_{n \to \infty} \sup a_n = \lim_{n \to \infty} \inf a_n = l$$

**and consequently $l$ is the only limit point of $\langle a_n \rangle$.**

(*b*) Let $< a_n >$ be defined as follows

$$a_1 = 1,\ a_{n+1} = \frac{4 + 3a_n}{3 + 2a_n}\ n \geq 1.$$

Show that $< a_n >$ converges. What is the limit of $< a_n >$ ?

## Section III

**5.** (*a*) State Maclaurin's Integral test. Show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n (\log n)^p}$$

converges if $p > 1$ and diverges if $0 < p \leq 1$.

(*b*) (*i*) A series $\Sigma\ u_n$ of positive terms converges iff there exists a number $K$ such that

$$u_1 + u_2 + \dots + u_n < K \quad \forall\ n.$$

(*ii*) Show that the series $1 + r + r^2 + \dots$ converges if $0 < r < 1$ and diverges if $r \geq 1$.

**6.** (*a*) Show that an absolutely convergent series is always convergent. Give an example to show that the converse is not always true.

(*b*) Prove that the series

$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \dots$$

converges.

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### B.A./B.Sc.(Hons.)/I – 2002

### MATHEMATICS – Unit IV

### (Analysis – II)

*Time : 2 hours* *Maximum Marks : 50*

*All questions are compulsory.*

*Attempt any* ***two*** *parts from each questions.*

**1.** (*a*) If $f$ be continuous on $[a, b]$ and $c$ be any real number between $f(a)$ and $f(b)$, then show that there exists a real number $x$ in $]a, b[$ such that $f(x) = c$.

(*b*) Let $$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational.} \end{cases}$$

Show that $f(x)$ is continuous only at $x = 1/2$.

(*c*) Evaluate :

(*i*) $\lim\limits_{x \to 0} \dfrac{\sin x - x + \frac{1}{6}x^3}{x^5}$ ; (*ii*) $\lim\limits_{x \to 0} \left( \dfrac{1}{e^x - 1} - \dfrac{1}{x} \right)$.

### B.A./B.Sc.(Hons.)/I – 2002
### MATHEMATICS – Unit III
### (Analysis – I)

*Time : 2 hours* *Maximum Marks : 50*

*Attempt **Four** questions in all, selecting only one question from each Section.*

### Section I

**1.** **(*a*) For all real numbers *x* and *y* show that : $|x + y| \le |x| + |y|$.**

**(*b*) (*i*) Let *S* be a non-empty subset of *R* such that sup *S* = inf *S*. What can be said about the set *S* ?**

**(*ii*) Give an example of a non-empty bounded subset *S* of *R* whose supremum and infimum both belong to *R* ~ *S*.**

**(*c*) State the order completeness property of real numbers. Show that any non-empty set of real numbers which is bounded below has an infimum.**

**2.** **(*a*) Show that a set $G \subset R$ is open if and only if for each $p \in G$, there exists a positive integer *n* such that**

$$\left] p - \frac{1}{n}, p + \frac{1}{n} \right[ \subset G .$$

**(*b*) Prove that a point *p* is a limit point of a set *S* iff every neighbourhood of *p* contains infinitely many points of *S*.**

**(*c*) If *M* and *N* are neighbourhoods of a point *p*, then $M \cap N$ is also a neighbourhood of *p*. Prove it.**

### Section II

**3.** **(*a*) Show that a monotone increasing sequence that is bounded above converges.**

**(*b*) Prove that the sequence $<a_n>$ defined by the relation**

$$a_1 = 1,\ a_n = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \ldots . + \frac{1}{(n-1)\,!},\ (n \ge 2)$$

**4.** (*a*) Show that a bounded sequence $\langle a_n \rangle \to l$ if and only if

$$\lim_{n \to \infty} \sup a_n = \lim_{n \to \infty} \inf a_n = l$$

and consequently *l* is the only limit point of $\langle a_n \rangle$.

(*b*) Let $< a_n >$ be defined as follows

$$a_1 = 1,\ a_{n+1} = \frac{4 + 3a_n}{3 + 2a_n}\ n \ge 1.$$

Show that $< a_n >$ converges. What is the limit of $< a_n >$ ?

### Section III

**5.** (*a*) State Maclaurin's Integral test. Show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}$$

converges if $p > 1$ and diverges if $0 < p \le 1$.

(*b*) (*i*) A series $\Sigma\ u_n$ of positive terms converges iff there exists a number *K* such that

$$u_1 + u_2 + \ldots . + u_n < K \quad \forall\ n.$$

(*ii*) Show that the series $1 + r + r^2 + \ldots$ converges if $0 < r < 1$ and diverges if $r \ge 1$.

**6.** (*a*) Show that an absolutely convergent series is always convergent. Give an example to show that the converse is not always true.

(*b*) Prove that the series

$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \ldots .$$

converges.

### B.A./B.Sc.(Hons.)/I – 2002
### MATHEMATICS – Unit IV
### (Analysis – II)

*Time : 2 hours* *Maximum Marks : 50*

*All questions are compulsory.*
*Attempt any **two** parts from each questions.*

**1.** (*a*) If *f* be continuous on $[a, b]$ and *c* be any real number between $f(a)$ and $f(b)$, then show that there exists a real number *x* in $]\,a, b\,[$ such that $f(x) = c$.

(*b*) Let $f(x) = \begin{cases} x & , \text{ if } x \text{ is rational} \\ 1 - x & , \text{ if } x \text{ is irrational.} \end{cases}$

Show that $f(x)$ is continuous only at $x = 1/2$.

(*c*) Define a uniformly continuous function on $[a, b]$. Show that the function $f(x) = 1/x^2 \ (x \neq 0)$ is uniformly continuous on $[a, \infty) \ (a > 0)$ but not uniformly continuous on $(0, \infty)$.

**2.** (*a*) State and prove Lagrange's mean value theorem. Prove that if $f$ is defined for all real $x$ such that $|f(x) - f(y)| < (x - y)^2$ for all $x, y \in R$, then $f$ is constant.

(*b*) Prove that : $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ whenever $0 < x < \pi/2$.

(*c*) Let $f$ and $g$ be differentiable functions in $[a, b]$ and let $g'(x) \neq 0$ for all $x \in [a, b]$. Prove that there exists some $c$ in $]a, b[$ such that

$$\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}.$$

**3.** (*a*) Assuming the validity of expansion, show that :

$$\sin(e^x - 1) = x + \frac{x^2}{2\,!} + \frac{5x^4}{4\,!} + \ldots..$$

(*b*) Examine the function $f(x)$ defined by

$$f(x) = |x|^m \,.\, |x - 1|^n \quad \forall \quad x \in R,$$

$m$ and $n$ being positive integers for extreme values. Also show that it has a maximum value

$$\frac{m^m \,.\, n^n}{(m + n)^{m+n}}.$$

(*c*) Evaluate :

(*i*) $\lim\limits_{x \to 0} \left( \cot^2 x - \dfrac{1}{x^2} \right)$

(*ii*) $\lim\limits_{x \to 0+0} (x^k \log x)$ (for each fixed positive real no. $k$).

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## B.A.(P)/III – 2002

## MATHEMATICS – Paper III (*ii*)

## (Real Analysis and Statistics)

*Time : 3 hours* *Maximum Marks : 100*

*All Sections are compulsory and have equal marks.*
*Attempt any **two** parts from each Section.*

## Part I – Real Analysis (*i*)

### Section I

**1.** (*a*) Define an open set and show that intersection of two open sets is open.

(*b*) Show that a point $x_0$ is a limit point of a set $S$ iff every *nbd*. of $x_0$ contains infinitely many points of $S$.

(*c*) Show that $f(x) = 8x^2$ is not uniformly continuous on $[0, \infty[$.

(*d*) If a function is defined on $[-1, 1]$ by :

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

then show that $f(x)$ is not continuous at any point.

### Section II

**2.** (*a*) Prove that : $\lim\limits_{n \to \infty} \left( \dfrac{n^n}{n\,!} \right)^{1/n}$ exists and equals $e$.

(*b*) Show that the sequence defined by

$$a_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots. + \frac{1}{2n - 1}$$

does not converge.

(*c*) State and prove Cauchy Integral Test for a positive term series.

(*d*) Show that the series :

$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} + \ldots\ldots$$

converges.

### Section III

**3.** (*a*) Obtain the Maclaurin's series expansion of $\cos x, x \in \mathbf{R}$.

(*b*) Evaluate :

(*i*) $\lim\limits_{x \to 0} \dfrac{x \cot x - 1}{x^2}$ ; (*ii*) $\lim\limits_{x \to \infty} \left( \dfrac{x^n}{e^x} \right), n > 0$.

(*c*) Show that $x < \sin^{-1} x < \dfrac{x}{\sqrt{1 - x^2}}, \ 0 < x < 1$.

(*d*) Find the maximum and minimum values of the function $f(x) = x + \dfrac{1}{x}, \ x \in \mathbf{R}, x \neq 0$.

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